Modeling, Control And Evaluation of Fluid Flow Systems Using Adaptation Based Linear Parameter Varying Models

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***Abstract*— In this paper a systematic method is proposed for the modeling and control of flow problems, as well as the evaluation of closed loop performance and robustness. A nonlinear Galerkin model representing the flow dynamics is approximated with an adaptation based linear parameter varying (LPV) system, and it is shown that the LPV model can represent the Galerkin model with acceptable error. This allows for one to use the simple LPV model in place of the more complex nonlinear Galerkin model for modeling the flow process. The error vector is interpreted as an external disturbance for the model and the parameter range suggested by the adaptation process is interpreted as the uncertainty range in which the the control design must perform satisfactorily. The approach developed is illustrated on a sample problem of controlling a flow governed by the Navier-Stokes equations.**

I. Introduction

The potential benefits offered by the ability to model and control fluid flow are vast, including reduced fuel costs for vehicles, and improved effectiveness of industrial processes; this makes flow control highly important from a technological viewpoint [1]. A few of the myriad studies on the topic include the control of channel flows [2], control of combustion instability [3], stabilization of bluff-body flow [4], control of cylinder wakes [5] and the control of cavity flows [6].

In this paper a systematic approach is built for the modeling and control design of flow problems, as well as the evaluation of closed loop robustness. An adaptation mechanism is built to yield a LPV model approximating the nonlinear Galerkin model representing the flow dynamics. It shown that, with proper choice of adaptation parameters, the LPV approximation can be made the represent the nonlinear Galerkin model with reasonably small error. This allows one to treat the LPV model as the actual flow model, with the error vector entering as a disturbance, and the parameter variations providing a range of uncertainty in which the control design must perform satisfactorily. The idea is illustrated through an example regarding flow on a square domain governed by Navier-Stokes equations.

II. Problem Description

The dynamics of flow processes are described by partial differential equations (PDEs) such as the Navier-Stokes Equations or Burgers' Equations, to which the control input enters through the boundary conditions. While these equations model the flow very accurately, they are quite complex, making it very difficult to use these PDEs directly for analysis and control design. For this reason these equations are commonly converted to low order Galerkin models, using standard methods such as Proper Orthogonal Decomposition (POD), Galerkin Projection (GP) [7], and extensions to these methods such as Input Separation techniques [8], [9]. The resulting model is termed a *Galerkin model* and has the following structure:

$$\begin{aligned}\dot{a}_i =& \sum_{k=1}^{n} L_{ik} a_k + \sum_{k=1}^{n} L_{\mathrm{in},ik}\gamma_k + \sum_{j=1}^{n}\sum_{k=1}^{n} Q_{ijk} a_k a_j \\ &+ \sum_{j=1}^{m}\sum_{k=1}^{n} Q_{\mathrm{ain},ijk} a_k \gamma_j + \sum_{j=1}^{m}\sum_{k=1}^{m} Q_{\mathrm{in},ijk}\gamma_k\gamma_j \qquad (1)\end{aligned}$$

for $i = 1, \dots, n$, where $a = \{a_i\}_{i=1}^{n} \in \mathbb{R}^n$ is the state vector and $\gamma = \{\gamma_i\}_{i=1}^{m} \in \mathbb{R}^m$ is the control input. The system above can be expressed in compact form as

$$\dot{a} = La + L_{\mathrm{in}}\gamma + Q(a,a) + Q_{\mathrm{ain}}(a,\gamma) + Q_{\mathrm{in}}(\gamma,\gamma) \qquad (2)$$

where $L = \{L_{ij}\}_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$, $L_{\mathrm{in}} = \{L_{\mathrm{in},ij}\}_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$, $Q(a,a) = \{a^T Q_i a\}_{i=1}^{n} \in \mathbb{R}^n$, $Q_i = \{Q_{ijk}\}_{j,k=1}^{n} \in \mathbb{R}^{n\times n}$, $Q_{\mathrm{ain}}(a,\gamma) = \{a^T Q_{\mathrm{ain},i}\gamma\}_{i=1}^{n} \in \mathbb{R}^n$, $Q_{\mathrm{ain},i} = \{Q_{\mathrm{ain},ijk}\}_{j,k=1}^{n,m} \in \mathbb{R}^{n\times m}$, $Q_{\mathrm{in}}(\gamma,\gamma) = \{a^T Q_{\mathrm{in},i} a\}_{i=1}^{n} \in \mathbb{R}^n$, $Q_{\mathrm{in},i} = \{Q_{\mathrm{in},ijk}\}_{j,k=1}^{n} \in \mathbb{R}^{m\times m}$. While the Galerkin system above is finite and of low order, it includes nonlinear terms both in the state a and the input γ, which present difficulties for analysis and control design. The approach followed in this paper aims at easing the analysis and control design process for this model, and can be summarized as follows: First a LPV system approximating system (2) is obtained, which has the following form

$$\dot{\hat{a}} = \hat{L}(\theta_L)\hat{a} + \hat{L}_{\mathrm{in}}(\theta_L)\gamma + \hat{L}_{\mathrm{err}}(\theta_L)e \qquad (3)$$

where $\theta_L = t \mapsto \theta_L(t)$ is the time-varying parameter vector and $e := \hat{a} - a$ is the error vector, which is desired to remain small and bounded as $t \to \infty$. The parameter vector θ_L is modified through an adaptation mechanism. The states of the system (3) will be called the *adapted states* or *reconstructed states*. After the LPV system in (3) is obtained, a nominal LTI system is extracted from this system, on which control design is performed using standard LTI control theory. The adaptation mechanism is used to determine an uncertainty range for the parameter vector θ_L, and the closed loop system is tested and evaluated for performance in this uncertainly range.

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III. Obtaining a LPV Model Approximating the Galerkin System Through Adaptation

In this section a LPV system of the form (3) is built to approximate the Galerkin system (2). First note that while system (1) is a nonlinear system in a and γ, it is linear in its parameter values contained in L, Q, L_{in}, Q_{in} and Q_{ain}. To write the dynamics in a form where this linear dependance is apparent, let us first build the parameter vector θ as

$$\theta := \operatorname{col}\left(L(:), L_{\text{in}}(:), Q(:), Q_{\text{in}}(:), Q_{\text{ain}}(:)\right) \tag{4}$$

where col stands for column vector, i.e. $\operatorname{col}\{x_1, x_2, \ldots, x_n\} = [x_1^T \;\; x_2^T \;\; \ldots \;\; x_n^T]^T$, and $L(:)$ denotes the column vector formed by stacking all elements of L on top of each other, i.e.

$$L(:) := \operatorname{col}\left(L_{11}, L_{21}, \ldots, L_{n1}, \ldots, L_{n1}, L_{n2}, \ldots L_{nn}\right) \;. \tag{5}$$

The column vectors $L_{\text{in}}(:)$, $Q(:)$, $Q_{\text{in}}(:)$ and $Q_{\text{ain}}(:)$ can be defined similarly. One can also define $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^p$ to satisfy the expression

$$\dot{a} = \Phi(a, \gamma)\theta \tag{6}$$

where $\Phi(a,\gamma)$ is a $n \times p$ dimensional matrix with elements $\{\Phi(a,\gamma)_{ij} \mid i = 1, \ldots, n, \; j = 1, \ldots, p\}$. Here, $\Phi(a,\gamma)_{ij}$ denotes the element at row i and column j and corresponds to the contribution of the jth parameter of θ to the ith element a. The next step is the design of the adaptation mechanism that will modify the system parameters. First, the Galerkin system (6) is written as

$$\begin{aligned}
\dot{a} &= \Phi(a,\gamma)\theta \\
\dot{a} &= \Phi_L(a,\gamma)\theta_L + \Phi_N(a,\gamma)\theta_N \\
\dot{a} &= La + L_{in}\gamma + Q(a,a) + Q_{ain}(a,\gamma) + Q_{in}(\gamma,\gamma)
\end{aligned} \tag{7}$$

where the linear and nonlinear parts of the system have been split as

$$\begin{aligned}
\Phi_L(a,\gamma)\theta_L &:= La + L_{in}\gamma \\
\Phi_N(a,\gamma)\theta_N &:= Q(a,a) + Q_{ain}(a,\gamma) + Q_{in}(\gamma,\gamma) \;.
\end{aligned} \tag{8}$$

We will also assume that $\|\operatorname{col}(a,\gamma)\| \leq \|\operatorname{col}(a,\gamma)\|_\infty$ for some $\|\operatorname{col}(a,\gamma)\|_\infty \in \mathbb{R}_+$ for all $t \geq 0$.[1] The goal is to obtain a linear model whose parameters will be modified by the adaptation mechanism to match the Galerkin system above. For this purpose one can construct a parameter adaptation mechanism in which the underlying model structure is based on only the linear part of the Galerkin model as follows

$$\dot{\hat{\theta}}_L = -\alpha\Phi_L^T(a,\gamma)e - \alpha\Upsilon_b(\hat{\theta}_L) \tag{9}$$

$$\dot{\hat{a}} = \Phi_L(a,\gamma)\hat{\theta}_L - ke = \hat{L}a + \hat{L}_{\text{in}}\gamma - ke \tag{10}$$

where $e = \hat{a} - a$, $k \in \mathbb{R}_+$, $\alpha \in \mathbb{R}_+$, and $b \in \mathbb{R}_+$. The function $\Upsilon_b : \mathbb{R}^n \to \mathbb{R}^n$ with $b \in \mathbb{R}_+$ is a dead-zone like function such that

$$\Upsilon_b(x) = \begin{cases} 0, & \|x\| \leq b; \\ x - bx/\|x\|, & \|x\| > b \;. \end{cases} \tag{11}$$

We now state the following theorem regarding the operation of the adaptation mechanism.

Theorem 1: Consider the system in (7) and adaptation mechanism given in (9)-(10). For this system and adaptation mechanism, the state error $e := \hat{a} - a$ and the error on the linear part of the adapted parameter vector $\tilde{\theta}_L := \hat{\theta}_L - \theta_L$ will remain bounded for all t, and $\exists k_1, k_2, k_3, k_4 \in \mathbb{R}_+$ and a class $\mathcal{N}$ function[2] ρ where

$$k_1 := \min\left\{\tfrac{1}{2}, \tfrac{1}{2\alpha}\right\} \tag{12}$$

$$k_2 := \max\left\{\tfrac{1}{2}, \tfrac{1}{2\alpha}\right\} \tag{13}$$

$$k_3 := \max\left\{\|Q\| + \tfrac{\|Q_{\text{ain}}\|}{2}, \tfrac{\|Q_{\text{ain}}\|}{2} + \|Q_{\text{in}}\|\right\} \tag{14}$$

$$k_4 := 3b \tag{15}$$

$$\rho(x) := \max\left\{\varepsilon^{-1}k_3x^2, k_4\right\} \tag{16}$$

such that

$$\|\operatorname{col}(e, \tilde{\theta}_L)\| \leq \sqrt{k_2/k_1}\,\rho(\|\operatorname{col}(a,u)\|_\infty) \tag{17}$$

as $t \to \infty$, for any $0 < \varepsilon < k$.

Proof: The proof is given in the Appendix. ■

It follows from the theorem that, while it is not possible to drive the error e to zero, it is possible to keep it acceptably small through proper selection of the internal parameters α and k of the adaptation mechanism. Note also that (10) can be manipulated as

$$\begin{aligned}
\dot{\hat{a}} &= \hat{L}(\hat{a} - e) + \hat{L}_{\text{in}}\gamma - ke \\
\dot{\hat{a}} &= \hat{L}\hat{a} + \hat{L}_{\text{in}}\gamma + \hat{L}_{\text{err}}e
\end{aligned} \tag{18}$$

where $\hat{L}_{\text{err}} = -(\hat{L} + kI)$. One can then see that (18) is of the same form as (3). Thus, one can now regard the system above as a LPV system that approximates the nonlinear Galerkin system (7), with the signal e entering as an external disturbance. With this interpretation, instead of dealing with a nonlinear system (i.e. the Galerkin system), one can carry out the control design on a linear system (e.g. a nominal linear time invariant (LTI) plant extracted from the LPV model) making sure that the controller also performs satisfactorily for the range of parameters predicted by the adaptation mechanism. Figure 1 shows a block diagram illustrating this idea for a tracking problem, with input and output disturbances also marked, in addition to the additional disturbance due to the adaptation error. The approach shown in the figure will be illustrated next with a flow control example governed by Navier-Stokes equations on a square domain.

[1] This assumption is justified from a physical perspective as for most real-life processes, the actuators can only produce finite inputs, and the system will not “blow up” under such inputs.

[2] Continuous, non-decreasing, and non-negative.

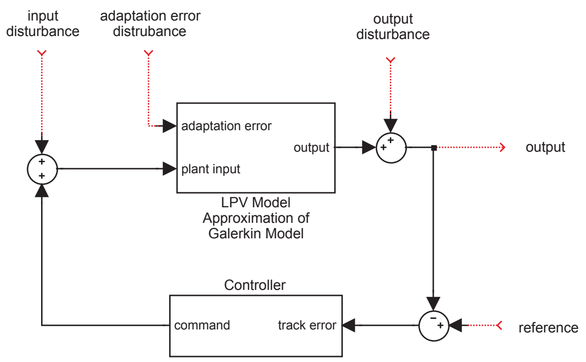

Fig. 1. Using a LPV model approximating the nonlinear Galerkin model for a reference tracking problem.

IV. Example: Boundary Control of 2D Incompressible Navier-Stokes Equations on a Square Domain

A. Problem Description

Consider the two-dimensional non-dimensional Navier-Stokes equation

$$\frac{\partial q}{\partial t} + (q \cdot \nabla)q = -\nabla p + \nu \Delta q \tag{19}$$

subject to the incompressibility condition $\nabla \cdot q = 0$, where $\nu = Re^{-1}$ and Re is the *Reynolds number*. Let $p(x, y, t) \in \mathbb{R}$ denote the pressure, and $q(x, y, t) = (u(x, y, t)\ v(u, x, t)) \in \mathbb{R}^2$ denote the flow velocity, where u and v are the components in the longitudinal and latitudinal directions, respectively. In the given coordinates, equation (19) reads as

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v &= -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v &= -\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{aligned} \tag{20}$$

For our example, the viscosity is set to $\nu = 0.1$, and the spatial domain is defined as $\Omega = [0, 1] \times [0, 1]$. The initial conditions are $u(x, y, 0) = v(x, y, 0) = 0$ and the boundary conditions are

$$\begin{aligned} &u(x, 0, t) = 1,\ \ v(x, 0, t) = 0 \\ &u(x, 1, t) = 1,\ \ v(x, 1, t) = 0 \\ &u(0, y, t) = 0,\ \ \frac{\partial v}{\partial x}(0, y, t) = 0 \\ &u(1, y, t) = \begin{cases} 0, & y \in [0, 0.42); \\ \gamma(t), & y \in [0.42, 0.58]; \\ 0, & y \in (0.58, 1]. \end{cases} \\ &v(1, y, t) = 0 \end{aligned}$$

where $\gamma \in \mathbb{R}$ is the control input. For this example, we shall define the control task as controlling the longitudinal speed at a given point $(x_c, y_c) \in \Omega$. In other words, if the system output y is defined as

$$y(t) = u(x_c, y_c, t) \tag{21}$$

and a reference signal $y_{\text{ref}} : t \mapsto y_{\text{ref}}(t)$ is given, then the goal is to achieve $y \to y_{\text{ref}}$.

B. Modeling of the Flow Process

It was shown in detail in [9] how a Galerkin system in the form (2) can be obtained from the Navier-Stokes equations (20) using proper orthogonal decomposition (POD), Galerkin Projection (GP) and input separation (IS). In this paper the results from this study will be used directly without repeating the derivations. However for control design, the system (2) must be augmented with an output equation. For this purpose an expanded POD expansion for (21) can be written as

$$\begin{aligned} y(t) &= u(x_c, y_c, t) = \sum_{i=1}^{n} a_i(t)\phi_{i,u}(x_c, y_c) + \gamma(t)\psi_u(x_c, y_c) \\ y &= L_{\text{out}}a + L_{\text{out,in}}\gamma\ . \end{aligned} \tag{22}$$

where ϕ_i are the POD baseline modes, ψ is the actuation mode, u is the flow in the longitudinal direction, $L_{\text{out}} := [\phi_{1,u}(x_c, y_c)\ \phi_{2,u}(x_c, y_c)\ \phi_{3,u}(x_c, y_c)] \in \mathbb{R}^{1\times 3}$ and $L_{\text{out,in}} := \psi_u(x_c, y_c) \in \mathbb{R}$.

C. Setting up the Adaptation Mechanism to Obtain the LPV Model

After the reduced order model is obtained, the next step is to build an adaptation mechanism as described in Section III. Recall that while the original Galerkin system (2) is nonlinear and contains quadratic terms, for the underlying model in estimation we wish to utilize only the linear portion of the model so as to produce a LPV model to be used for the control design process at the next stage. To reduce the number parameters in the adaptation process and simplify the calculations it will be assumed that the Galerkin system has been transformed into modal form. In this case, it was shown in [9] that the eigenvalues for L are of the form $\text{spec}(L) = \{\lambda_1, \lambda_2, \lambda_3\}$ and therefore L and L_{in} in modal form are

$$L = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},\ L_{\text{in}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{23}$$

where $\lambda_1, \lambda_2, \lambda_3, b_1, b_2, b_3 \in \mathbb{R}$. Define $\hat{\theta}_L := [\hat{\lambda}_1\ \hat{\lambda}_2\ \hat{\lambda}_3\ \hat{b}_1\ \hat{b}_2\ \hat{b}_3]^T$ and

$$\Phi_L(a, u) := \begin{bmatrix} a_1 & 0 & 0 & \gamma & 0 & 0 \\ 0 & a_2 & 0 & 0 & \gamma & 0 \\ 0 & 0 & a_3 & 0 & 0 & \gamma \end{bmatrix}.$$

With these definitions, we set up the parameter adaptation mechanism (9) and the LPV model (10) whose parameters are varied by this adaptation mechanism. Recall that the main goal of the adaptation mechanism is to provide an estimate for the range Θ in which the parameters of the LPV system will vary. For this purpose a high number input signals of various types we applied to the Galerkin system including ramp functions, sine functions, chirp functions, square waves and white noise, and the values assumed by the parameters under these excitation signals were recorded.

Observing the range in which the parameter values vary with these excitations, the range Θ such that $\hat{\theta}_L \in \Theta$ is chosen to be the 6-dimensional box

$$\begin{aligned}\Theta = \{\hat{\theta}_L \in \mathbb{R}^6 \ : \ &-209.83 < \hat{\lambda}_1 < -139.89, \\ &-10.79 < \hat{\lambda}_2 < -7.19, -40.10 < \hat{\lambda}_3 < -26.73, \\ &-28.90 < \hat{b}_1 < -2.72, -5.36 < \hat{b}_2 < -0.73, \\ &-5.24 < \hat{b}_3 < -1.20\} \ . \end{aligned} \tag{24}$$

Recall also that the adaptation system has two design parameters, α and k, that determine the rate of convergence of the scheme and also the bound on the error $e = \hat{a} - a$. (cf. Theorem 1). Based also on the various types of inputs mentioned above, these parameters were chosen as $k = 1000$ and $\alpha = 100$ which were seen to yield $\|e\| < 10^{-3}$ for all excitation cases mentioned above.

D. Controller Design and Evaluation

Once the range Θ for the parameter vector is known as in (24), the control design and evaluation is carried out in the following steps: 1) Extract a nominal LTI model from the LPV system, 2) Design a controller for the nominal model using standard LTI design methods (e.g. LQR, PI, PID, etc.), 3) Assure that controller designed performs satisfactorily over the entire parameter range Θ. To obtain the LTI nominal plant for the first step, the parameters are simply fixed at some point near the center of the parameter range Θ in (24) as follows: $\hat{\theta}_{Ln} := \text{col}\{-175, -8.99, -33.4, -15.81, -3.05, -3.22\}$, which yields the nominal LTI plant

$$\begin{aligned}\dot{\hat{a}} &= \hat{L}(\hat{\theta}_{Ln})\hat{a} + \hat{L}_{\text{in}}(\hat{\theta}_{Ln})\gamma + \hat{L}_{\text{err}}(\hat{\theta}_{Ln})e \\ y &= L_{\text{out}}a + L_{\text{out,in}}\gamma \ . \end{aligned} \tag{25}$$

where the parameter dependency of $\hat{L}$, $\hat{L}_{\text{in}}$ and $\hat{L}_{\text{err}}$ have been shown explicitly.[3] For step two, we choose a simple PI controller for the nominal plant in (25) as follows

$$C(s) := 5 + \frac{500}{s} \tag{26}$$

where $\Gamma(s) = C(s)E_r(s)$, Γ is the transfer function of the input γ, and E_r is that of $e_r := y_{\text{ref}} - y$, the tracking error. The third step is the verification that the controller performs satisfactorily for the entire parameter range Θ. Figure 2 shows the closed loop step response of the system from the reference y_{ref} to output y for the nominal plant and ten random values of the parameter vector $\hat{\theta}_L$. It can be seen that the closed loop system is successful is tracking the step reference in all cases. Figure 3 shows the closed loop step response from the adaptation error e to out to output y for the nominal plant and ten random values of the parameter vector $\hat{\theta}_L$. Recall that the adaptation error $e := \text{col}\{e_1, e_2, e_3\}$ is regarded as a disturbance for the system, as seen in Figure 1. It can be seen that a step error does not cause instability and the worst amplification is from e_2 to y, which is about

[3]Recall that L_{out} and $L_{\text{out,in}}$ are not parameter dependent. Recall also that the adaptation error e is treated as a disturbance

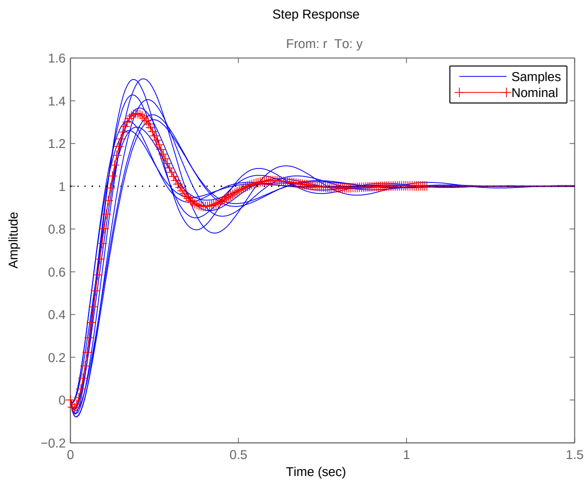

Fig. 2. Step response of the closed loop system from reference y_{ref} to output y for ten random values of the parameter vector.

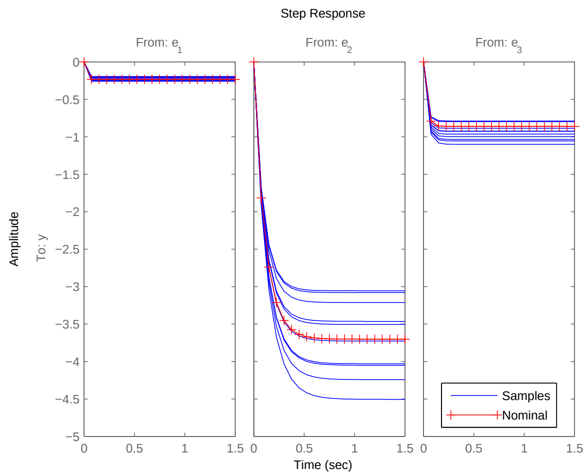

Fig. 3. Step response of the closed loop system from the adaptation error e to output y for ten random values of the parameter vector.

five times. As mentioned in Section IV-C, the adaptation parameters k and α were chosen so as to limit the the error amplitude to $\|e\| < 10^{-3}$; hence a five-time amplification will yield to an error contribution of at most 0.005 at the output, which is acceptable. The parameter range Θ provided by the adaptation mechanism is also useful in determining the sensitivity to input and output disturbances, entering the system as shown in Figure 1. Figure 4 shows the step response and Figure 5 shows the frequency response of the closed loop system to an input disturbance for the nominal plant and ten random values of the parameter vector. The value of the parameter vector yielding to the highest peak in the frequency response and its corresponding step response are also shown with dashed lines in the figures. The figures indicate that the closed loop system in general has good input disturbance rejection properties, but one needs to be careful if input noise of high amplitude around 15 rad/s is expected since its attenuation may be slow and may interfere with the

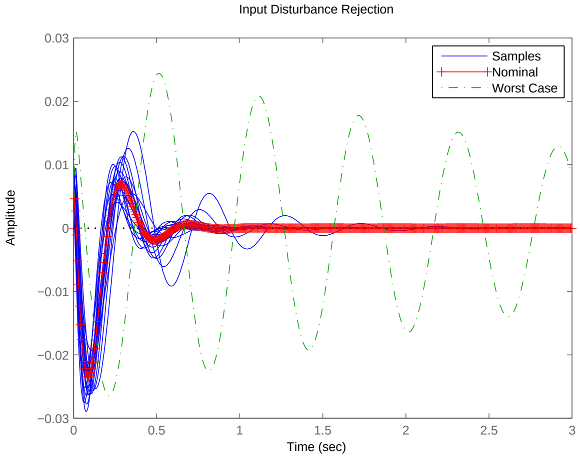

Fig. 4. Response of the closed loop system to a step input disturbance for ten random values of the parameter vector.

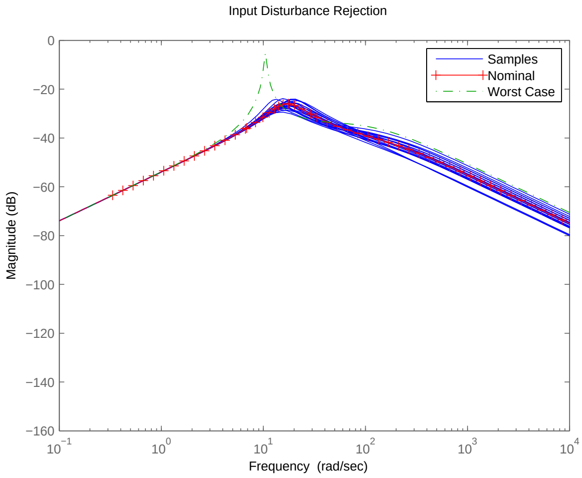

Fig. 5. Frequency response of the closed loop system to input disturbance for ten random values of the parameter vector.

command signal given from the controller. A similar analysis can be carried out for the case of output disturbance.

The next step is to connect the controller in feedback with the actual Navier-Stokes equations (20) and perform computational fluid dynamics (CFD) simulations to evaluate the performance of the closed loop system. For this purpose *Navier2d* solver under MATLAB [10] was utilized, with the reference signal y_{ref} being a step signal dropping from 0.5 to -0.5 at $t = 0.7$ seconds. In addition, an input noise of $0.5\sin(15t)$ was also applied to the system; recall from Figure 5 that $\omega = 15$ rad/s is the frequency under which the frequency response peaks for the worst case. The snapshots for the CFD simulation for this case are shown in Figures 6-7 and the system output (21), i.e. the u-velocity at the center of the domain, is shown in Figure 8. It can be observed that the closed loop system is successful in tracking the input reference, while at the same time significantly attenuating the effect of the input disturbance to the output, which is

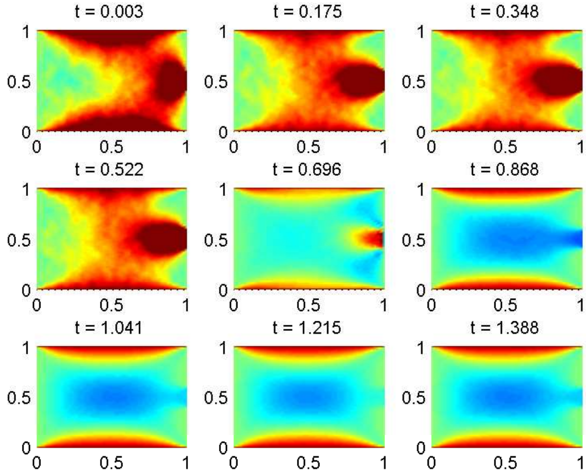

Fig. 6. Snapshots (u-component) of CFD simulation for the Navier-Stokes system under closed loop with input noise.

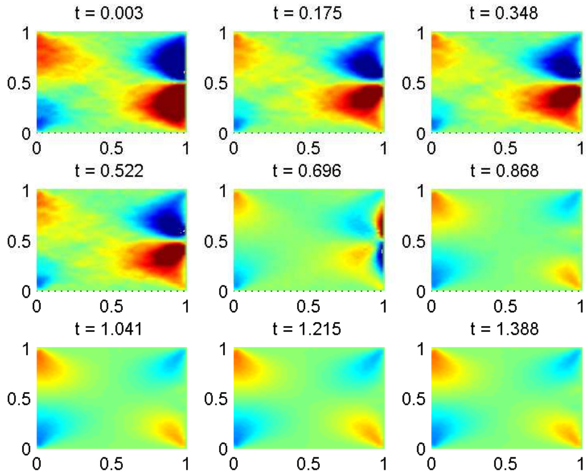

Fig. 7. Snapshots (v-component) of CFD simulation for the Navier-Stokes system under closed loop with input noise.

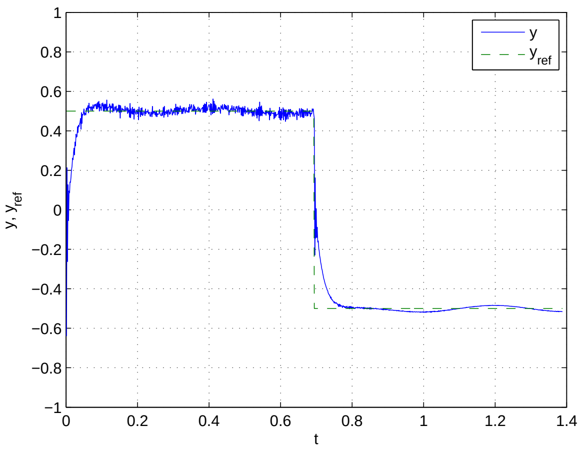

Fig. 8. Point of interest (i.e. system output y) for the Navier-Stokes system under closed loop with input noise.

consistent with the earlier analysis based on the LPV model approximation.

V. Conclusions and Future Works

In this paper we considered a systematic approach to the modeling and control design of flow problems, as well as the evaluation of closed loop performance and robustness. An adaptation scheme was then built to yield a LPV model approximating the nonlinear Galerkin model representing the flow and it was shown that the LPV approximation can be made to represent the nonlinear Galerkin model with reasonably small error. This allowed for the treatment of the LPV model as the actual flow model, with the error vector entering as a disturbance, and the parameter variations providing a range of uncertainty in which the control design must perform satisfactorily. The idea was demonstrated through a Navier-Stokes flow control example on a square domain, where the control goal was to make the center point longitudinal velocity track a given reference.

Our current and future research directions include utilizing the parameters from the adaptation scheme in the controller design (e.g. scheduling techniques), obtaining LPV model approximations for different reduced order models and testing current and new approaches on other flow control case studies.

Appendix

Proof of Theorem 1

To analyze the stability and boundedness of (8)-(10) we first define a Lyapunov-like function

$$V(e,\tilde{\theta}_L) = \frac{1}{2}\|e\|^2 + \frac{1}{2\alpha}\|\tilde{\theta}_L\|^2 \ .$$

This function is positive definite and quadratic, and can be bounded from the above and below by the positive definite functions

$$W_1\left(\mathrm{col}(e,\tilde{\theta}_L)\right) := k_1\|\mathrm{col}(e,\tilde{\theta}_L)\|^2 \tag{27}$$

$$W_2\left(\mathrm{col}(e,\tilde{\theta}_L)\right) := k_2\|\mathrm{col}(e,\tilde{\theta}_L)\|^2 \tag{28}$$

with $k_1 := \min\{\frac{1}{2}, \frac{1}{2\alpha}\}$ and $k_2 := \max\{\frac{1}{2}, \frac{1}{2\alpha}\}$. Differentiating V along the trajectories of the system (9)-(10) yields

$$\begin{aligned}\dot{V} &= e^T\dot{e} + \frac{1}{\alpha}\tilde{\theta}_L^T\dot{\tilde{\theta}}_L = \ e^T(\dot{\hat{a}} - \dot{a}) + \frac{1}{\alpha}\tilde{\theta}_L^T\dot{\tilde{\theta}}_L \\ &\leq -k\|e\|^2 + \|e\|\|Q\|\|a\|^2 + \|e\|\|Q_{\mathrm{ain}}\|\|a\|\|\gamma\| \\ &\quad + \|e\|\|Q_{\mathrm{in}}\|\|\gamma\|^2 - \Upsilon_b^T(\tilde{\theta}_L + \theta_L)\tilde{\theta}_L \ .\end{aligned}$$

Pick any ε such that $0 < \varepsilon < k$. Then

$$\begin{aligned}\dot{V} \leq &-(k-\varepsilon)\|e\|^2 + \|e\|\Big(-\varepsilon\|e\| + \|Q\|\|a\|^2 \\ &+ \|Q_{\mathrm{ain}}\|\|a\|\|\gamma\| + \|Q_{\mathrm{in}}\|\|\gamma\|^2\Big) \\ &- \Upsilon_b^T(\tilde{\theta}_L + \theta_L)\tilde{\theta}_L \\ &+ \|e\|\|Q_{\mathrm{in}}\|\|\gamma\|^2 - \Upsilon_b^T(\tilde{\theta}_L + \theta_L)\tilde{\theta}_L \\ \leq &-(k-\varepsilon)\|e\|^2 + \|e\|\left(-\varepsilon\|e\| + k_3\|\mathrm{col}(a,u)\|^2\right) \\ &- \Upsilon_b^T(\tilde{\theta}_L + \theta_L)\tilde{\theta}_L\end{aligned} \tag{29}$$

where $k_3 := \max\{\|Q\| + \frac{\|Q_{\mathrm{ain}}\|}{2}, \frac{\|Q_{\mathrm{ain}}\|}{2} + \|Q_{\mathrm{in}}\|\}$, and we have used Young's inequality as needed.[4] The first term $-(k-\varepsilon)\|e\|^2$ is clearly non-positive. The second term $\|e\|\left(-\varepsilon\|e\| + k_3\|\mathrm{col}(a,u)\|^2\right)$ will be less than or equal to zero if $\|e\| > \varepsilon^{-1}k_3\|\mathrm{col}(a,\gamma)\|^2$. To show the non-positiveness of the third term $-\Upsilon_b^T(\tilde{\theta}_L + \theta_L)\tilde{\theta}_L = -\tilde{\theta}_L^T\Upsilon_b(\tilde{\theta}_L + \theta_L)$, one can easily prove that for the dead-zone function Υ_b, there exist $k_4, k_5 \in \mathbb{R}_+4$ such that for $\|\tilde{\theta}_L\| > k_4$ it holds that $\tilde{\theta}_L^T\Upsilon_b(\tilde{\theta}_L + \theta_L) \geq k_5\|\tilde{\theta}_L\|^2$. Thus, if it is the case that $\|\tilde{\theta}_L\| > k_4$ then it holds that $-\tilde{\theta}_L^T\Upsilon_b(\tilde{\theta}_L + \theta_L) \leq -k_5\|\tilde{\theta}_L\|^2 \leq 0$ and hence the third term (29) will be less than or equal to zero. Then, for the entire state vector $\mathrm{col}(e,\tilde{\theta}_L)$, if

$$\begin{aligned}\|\mathrm{col}(e,\tilde{\theta}_L)\| > &\max\{\varepsilon^{-1}k_3\|\mathrm{col}(a,\gamma)\|_\infty^2, k_4\} \\ &=: \rho(\|\mathrm{col}(a,\gamma)\|_\infty) := \mu\end{aligned} \tag{30}$$

then it is clear that

$$\begin{aligned}\dot{V} \leq &-(k-\varepsilon)\|e\|^2 + \|e\|\left(-\varepsilon\|e\| + k_3\|\mathrm{col}(a,u)\|^2\right) \\ &- \Upsilon_b^T(\tilde{\theta}_L + \theta_L)\tilde{\theta}_L \leq 0 \\ \leq &-(k-\varepsilon)\|e\|^2 - k_5\|\tilde{\theta}_L\|^2 \\ \leq &-k_6\|\mathrm{col}(e,\tilde{\theta}_L)\|^2 := -W_3(\mathrm{col}(e,\tilde{\theta}_L)) < 0\end{aligned}$$

where $k_6 := \min\{k-\varepsilon, k_5\} > 0$. Thus using (27), (28) and (30), it can be seen that for $t \to \infty$ we have

$$\begin{aligned}W_1\left((\mathrm{col}(e,\tilde{\theta}_L)\right) &\leq \max_{\|\mathrm{col}(e,\tilde{\theta}_L)\| \leq \mu} W_2\left(\mathrm{col}(e,\tilde{\theta}_L)\right) \\ \|\mathrm{col}(e,\tilde{\theta}_L)\| &\leq \sqrt{k_2/k_1}\,\rho(\|\mathrm{col}(a,\gamma)\|_\infty)\end{aligned}$$

which is the statement of the theorem.

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[4]If $x, y \in \mathbb{R}_+$ then $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$.