Proceedings of the 2007 American Control Conference
Marriott Marquis Hotel at Times Square
New York City, USA, July 11-13, 2007

Analysis and Nonlinear Control of Galerkin Models Using Averaging and Center Manifold Theory

Coşku Kasnakoğlu and Andrea Serrani

Abstract— **In this paper, nonlinear control systems whose dynamics are quadratic with respect to state, and bilinear with respect to state and input, which exhibit an oscillation caused by a stable limit cycle for zero input are studied. The effect of linear control on this model is analyzed using modal forms and center manifold theory. It is found that the oscillation amplitude depends both on terms linear in the control and those that depend on the center manifold. To exploit the latter, a nonlinear control law is proposed. The closed loop system is simplified using a time varying periodic change of coordinates, time scaling, and averaging. Using center manifold theory, conditions governing the number and stability type of the limit cycles, and analytical expressions for the oscillation amplitude are derived. The results are verified using a finite dimensional cavity flow model as an example.**

Index Terms— **Galerkin projection, proper orthogonal decomposition, center manifold theory, averaging theory, reduced order modeling, cavity flow control, Navier-Stokes equation**

I. INTRODUCTION

Systems in many areas and applications are described by dynamics which are quadratic with respect to the state and bilinear with respect to the input and the state. A system model of this type is the Galerkin model, which has been utilized extensively in the field of flow control, e.g. [9], [13], [3]. The work presented here deals with the simplification, analysis and control of a system described by a Galerkin model, by means of center manifold theory [14], which is an important tool in nonlinear system theory. Work using center manifold theory includes [1], [7], [2].

The goal of this paper is to start with a Galerkin model, which is a representative of a class of systems that frequently arise in flow control problems, and apply center manifold theory, together with averaging, for the analysis, simplification and control of this model. The paper is organized as follows: The description of the problem is given in Section II. Classical linear control approach to the problem is analyzed in Section III, followed by model reduction, nonlinear control design and analysis in Section IV. The results are applied to a cavity flow control problem in Section V and numerical simulation results are presented in Section VI. Conclusions and future work are given in Section VII.

This work is supported in part by AFOSR and AFRL/VA through the Collaborative Center of Control Science (Contract F33615-01-2-3154).

The authors are with the Department of Electrical and Computer Engineering, The Ohio State University, Columbus, OH 43210, USA. Corresponding author: Coşku Kasnakoğlu (kasnakoglu.1@osu.edu)

II. PROBLEM DESCRIPTION

Consider the following N-dimensional nonlinear control system

$$\dot{a}_i = \sum_{j=1}^{N} l_{ij} a_j + \sum_{j,k=1}^{N} q_{ijk} a_j a_k + \left(r_i + \sum_{j=0}^{N} s_{ij} a_j \right) u \quad (1)$$

with $i = 1 \dots N$. System (1) can be expressed in compact form as

$$\dot{a} = La + Q(a) + (R + Sa)\, u$$

where $a = \{a_i\}_{i=1}^N \in \mathbb{R}^N$, $u \in \mathbb{R}$, $L = \{l_{ij}\}_{i,j=1}^N \in \mathbb{R}^{N\times N}$, $Q(a) = \{a^T Q_i a\}_{i=1}^N \in \mathbb{R}^N$, $Q_i = \{q_{ijk}\}_{j,k=1}^N \in \mathbb{R}^{N\times N}$, $R = \{r_i\}_{i=1}^N \in \mathbb{R}^N$ and $S = \{s_{ij}\}_{i,j=1}^N \in \mathbb{R}^{N\times N}$.

Assumption 1. *System (1), when $u = 0$, has a stable limit cycle. Furthermore, it exhibits a locally oscillatory behavior described by the eigenvalue spectrum of L as $\mathtt{spec}(L) = \{\sigma + j\omega, \sigma - j\omega, -\lambda_1, \cdots, -\lambda_N\}$ where $\sigma > 0$, $\omega > 0$, $\lambda_i > 0$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. This structure will always be preserved, even in closed loop.*

Using a non-singular transformation, system (1) can be represented in modal coordinates as

$$\begin{aligned} \dot{\eta} &= F_1\eta + \varphi_1(\eta,\zeta) + (G_1 + \gamma_1(\eta,\zeta))u \\ \zeta &= F_2\eta + \varphi_2(\eta,\zeta) + (G_2 + \gamma_2(\eta,\zeta))u \end{aligned} \quad (2)$$

where

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix},\ \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{N-2} \end{bmatrix},\ F_1 = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix}$$

and $F_2 = \mathtt{diag}(-\lambda_1, \cdots, -\lambda_N)$. Also $G_1 \in \mathbb{R}^2$, $G_2 \in \mathbb{R}^{N-2}$, and $\varphi_1, \gamma_1 : \mathbb{R}^2 \times \mathbb{R}^{N-2} \to \mathbb{R}^2$ and $\varphi_2, \gamma_2 : \mathbb{R}^2 \times \mathbb{R}^{N-2} \to \mathbb{R}^{N-2}$ are continuously differentiable functions which vanish at the origin together with their first derivatives.

Model (2) represents a class of systems obtained when one considers a reduced order model for flow control problems (e.g. [9], [11], [12], [10]).

III. ANALYSIS FOR LINEAR PART OF CONTROL

For the sake of simplicity, without loss of generality, the analysis will be restricted to the case $N = 4$; it should be clear how to extend the results in case there are additional stable modes. Consider a control law of the form

$$u = K\eta + \bar{K}(\eta)$$

where $\bar{K}(\eta) = \mathcal{O}(\eta_1^2 + \eta_2^2) = \mathcal{O}(\rho^2)$, being $\rho = \sqrt{\eta_1^2 + \eta_2^2}$ and $\theta = \arctan(\eta_2/\eta_1)$ the polar coordinate representation of η. The closed loop system under the given control law is given by

$$\begin{aligned} \dot{\eta} &= (F_1 + G_1 K)\eta + \varphi(\eta,\zeta) + \gamma_1(\eta,\zeta)K\eta \\ &\quad +(G_1 + \gamma_1(\eta,\zeta))\bar{K}(\eta) \\ \dot{\zeta} &= F_2\zeta + G_2 K\eta + \varphi(\eta,\zeta) + \gamma_2(\eta,\zeta)K\eta \\ &\quad +(G_2 + \gamma_2(\eta,\zeta))\bar{K}(\eta) \end{aligned} \tag{3}$$

where

$$F_1 + G_1 K = \begin{bmatrix} \sigma + g_{1,1}K_1 & -\omega + g_{1,1}K_2 \\ \omega + g_{1,2}K_1 & \sigma + g_{1,2}K_2 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix},\ G_2 = \begin{bmatrix} g_{21} \\ g_{22} \end{bmatrix}$$

and $K = [K_1\ K_2]$. Transforming the above system into modal form yields

$$\begin{aligned} \dot{\eta} &= \bar{F}_1(K)\eta + \bar{\Phi}_1(\eta,\zeta,K) \\ \dot{\zeta} &= F_2\zeta + \bar{\Phi}_2(\eta,\zeta,K) \end{aligned} \tag{4}$$

where

$$\bar{F}_1(K) = \begin{bmatrix} \bar{\sigma}(K) & -\bar{\omega}(K) \\ \bar{\omega}(K) & \bar{\sigma}(K) \end{bmatrix}$$

and $\bar{\Phi}_1, \bar{\Phi}_2 : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ collect the nonlinear terms. In addition

$$\begin{aligned} \bar{\sigma}(K) &= \sigma + 1/2\, g_{1,2}K_2 + 1/2\, g_{1,1}K_1 \\ \bar{\omega}(K) &= \frac{1}{2}(-g_{1,2}^2 {K_2}^2 - 2\, g_{1,2}K_2 g_{1,1}K_1 - g_{1,1}^2 {K_1}^2 \\ &\quad +4\,\omega^2 - 4\,\omega\, g_{1,1}K_2 + 4\, g_{1,2}K_1\omega)^{1/2}\ . \end{aligned}$$

Regarding $\bar{\sigma}$ as a state obeying a trivial dynamics, it is possible to apply center manifold theory to analyze the local behavior of the trajectories of the system in (4). Converting the system to polar coordinates $\dot{\rho} = \bar{\sigma}\rho + \mathcal{O}(\rho^2)$, one can approximate the center manifold as an expansion of the form $\zeta = h(\rho,\bar{\sigma}) = \alpha_1(K)\rho^2 + \alpha_2(K)\bar{\sigma}^2 + \alpha_3(K)\rho\bar{\sigma} + \mathcal{O}(3)$. Substituting this into the homology equation

$$\frac{\partial h}{\partial \rho}(\rho,\bar{\sigma})\left(\bar{\sigma}\rho + \mathcal{O}(\rho^2)\right) = F_2 h(\rho,\bar{\sigma}) + \bar{\Phi}_2(\rho,\theta,h(\rho,\bar{\sigma}),K)$$

and solving for α_i yields $\alpha_2 = \alpha_3 = 0$. Thus an approximation of the least nontrivial order for the center manifold is of the form $\zeta = h(\rho) = \alpha_1(K)\rho^2 + \texttt{h.o.t.}$; making this substitution and transforming into polar coordinates one obtains the reduced dynamics

$$\begin{aligned} \dot{\rho} &= (\bar{\sigma}(K) - \bar{\alpha}(K)\rho^2)\rho + \texttt{h.o.t.} \\ \dot{\theta} &= \omega + \bar{\beta}(K)\rho^2 + \texttt{h.o.t.} \end{aligned} \tag{5}$$

where $\bar{\alpha}(K)$ and $\bar{\beta}(K)$ are parameters depending on the center manifold and h.o.t. stands for 'higher order terms'. The fact that $\bar{\sigma}(0) > 0$ and $\bar{\alpha}(0) > 0$ follow from the Assumption 1. From (5) one sees that, if $\bar{\sigma}(K) < 0$ and $\bar{\alpha}(K) > 0$, then $\rho = 0$ is asymptotically stable. If $\bar{\sigma}(K) < 0$ and $\bar{\alpha}(K) < 0$ then $\rho = 0$ is unstable and the system also has an unstable limit cycle. If $\bar{\sigma}(K) > 0$ and $\bar{\alpha}(K) < 0$ then $\rho = 0$ is unstable and the system has no limit cycle. The interesting case is when we have $\bar{\sigma}(K) > 0$ and $\bar{\alpha}(K) > 0$; for this case $\rho = 0$ is unstable and the system has a stable oscillation, with amplitude and frequency

$$\rho^* = \sqrt{\frac{\bar{\sigma}(K)}{\bar{\alpha}(K)}}\ , \qquad \omega^* = \omega + \beta(K)\frac{\bar{\sigma}(K)}{\bar{\alpha}(K)} \tag{6}$$

Recall that $\bar{\sigma}(K) = \sigma + 1/2\, g_{2,2}K_2 + 1/2\, g_{1,1}K_1$ and $\bar{\alpha}(K)$ depends on the center manifold. One sees from (6) that there are two ways of decreasing ρ^* by feedback: the first is to decrease $\bar{\sigma}(K)$ by means of K, and the second is to increase $\bar{\alpha}(K)$. For the former way, some examples of flow control are [8], [11], [15], [4]. The goal of this paper is to explore the latter way, and provide an analysis on the effect of the nonlinear part of the control. Towards this goal, first averaging theory [6] will be used to simplify the system and obtain more structure to exploit in the analysis.

IV. MODEL REDUCTION AND ANALYSIS FOR THE NONLINEAR PART

For the system in (2), consider a parametrized family of control laws of the form $u(\eta, K)$, where u is smooth, with $u(0,K) = 0$ and $\left.\frac{\partial u}{\partial \eta}\right|_{\eta=0} = 0$. The closed loop system is written as

$$\begin{aligned} \dot{\eta} &= F_1\eta + \varphi_1(\eta,\zeta) \\ &\quad +(G_1 + \gamma_1(\eta,\zeta))u(K,\eta) =: f_\eta(\eta,\zeta) \\ \zeta &= F_2\zeta + \varphi_2(\eta,\zeta) \\ &\quad +(G_2 + \gamma_2(\eta,\zeta))u(K,\eta) =: f_\zeta(\eta,\zeta) \end{aligned} \tag{7}$$

Define a time-varying periodic change of coordinates $\eta^\vartheta = R(\vartheta)\eta$ where

$$R(\vartheta) = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}\ , \quad \vartheta = \omega_c t$$

and ω_c is the frequency of oscillation of the limit cycle, as given by ω^* in (6). Notice that in the above ϑ can be interpreted as a new time scale. Using the above transformation one gets

$$\begin{aligned} \dot{\eta}^\vartheta &= R(\vartheta)f_\eta(R^T(\vartheta)\eta^\vartheta,\zeta) + \dot{R}(\vartheta)R^T(\vartheta)\eta^\vartheta =: f_\eta^\vartheta(\vartheta,\eta^\vartheta,\zeta) \\ \dot{\zeta} &= f_\zeta(R^T(\vartheta)\eta^\vartheta,\zeta) =: f_\zeta^\vartheta(\vartheta,\eta^\vartheta,\zeta)\ . \end{aligned} \tag{8}$$

Note that one can also view this change of coordinates as approximating the solution with its first harmonic, with time-varying coefficients, and then looking at the dynamics of these coefficients. This is indeed closely related to the Krylov-Bogoliubov averaging (see [5] for details). Let $\epsilon = \omega_c^{-1}$ and write

$$\begin{aligned} \frac{d\eta^\vartheta}{d\vartheta} &= \epsilon f_\eta^\vartheta(\vartheta,\eta^\vartheta,\zeta^\vartheta) \\ \frac{d\zeta^\vartheta}{d\vartheta} &= \epsilon f_\zeta^\vartheta(\vartheta,\eta^\vartheta,\zeta^\vartheta) \end{aligned} \tag{9}$$

Note that f_η^ϑ and f_ζ^ϑ are bounded with respect to ϑ and ϵ since these only appear in $\sin$ and $\cos$ functions. Viewing ϑ

as the new time variable one can average over $\vartheta \in [0, 2\pi]$ with states $(\bar{\eta}^{\vartheta}, \bar{\zeta})$ as

$$\frac{d\bar{\eta}^{\vartheta}}{d\vartheta} = \frac{\epsilon}{2\pi}\int_0^{2\pi} f_{\eta}^{\vartheta}(\vartheta, \bar{\eta}^{\vartheta}, \bar{\zeta})\, d\vartheta \ =: \ \epsilon\, f_{\eta,\mathrm{avg}}^{\vartheta}(\bar{\eta}^{\vartheta}, \bar{\zeta})$$
$$\frac{d\bar{\zeta}}{d\vartheta} = \frac{\epsilon}{2\pi}\int_0^{2\pi} f_{\zeta}^{\vartheta}(\vartheta, \bar{\eta}^{\vartheta}, \bar{\zeta})\, d\vartheta \ =: \ \epsilon\, f_{\zeta,\mathrm{avg}}^{\vartheta}(\bar{\eta}^{\vartheta}, \bar{\zeta})$$

which, switching back to the original coordinates and time scale, and dropping bars for ease of notation, yields an averaged system for (η, ζ) the form

$$\begin{aligned} \dot{\eta} &= f_{\eta,\mathrm{avg}}(\eta, \zeta) \\ \dot{\zeta} &= f_{\zeta,\mathrm{avg}}(\eta, \zeta)\ . \end{aligned} \tag{10}$$

Remember that $f_{\eta,\mathrm{avg}}$ and $f_{\eta,\mathrm{avg}}$ above depend on the input $u(\eta, K)$ implicitly.

As mentioned in the previous section, the goal is to single out the effect of the linear part of the control, i.e. in the dynamics for η above it is desirable for the input to enter only nonlinearly. It can be shown that any phase invariant control, i.e. one of the form $u = u(\rho, K)$ will achieve this effect, resulting in an averaged dynamics of the form:

$$\begin{aligned} \begin{bmatrix} \dot{\eta} \\ \dot{\zeta} \end{bmatrix} &= \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} \eta \\ \zeta \end{bmatrix} + \begin{bmatrix} \Phi_1(\eta,\zeta) \\ \Phi_2(\eta,\zeta) \end{bmatrix} \\ &\quad + \left(\begin{bmatrix} 0 \\ B_2 \end{bmatrix} + \begin{bmatrix} g_1(\eta) \\ g_2(\zeta) \end{bmatrix} \right) u(\rho, K) \end{aligned} \tag{11}$$

where

$$\Phi_1(\eta,\zeta) = \begin{bmatrix} \phi_{11}\eta_1\zeta_1 + \phi_{12}\eta_1\zeta_2 + \phi_{21}\eta_2\zeta_1 + \phi_{22}\eta_2\zeta_2 \\ -\phi_{21}\eta_1\zeta_1 - \phi_{22}\eta_1\zeta_2 + \phi_{11}\eta_2\zeta_1 + \phi_{12}\eta_2\zeta_2 \end{bmatrix}$$
$$\Phi_2(\eta,\zeta) = \begin{bmatrix} \phi_{31}(\eta_1^2+\eta_2^2) + \phi_{32}\zeta_1^2 + \phi_{33}\zeta_2^2 + \phi_{34}\zeta_1\zeta_2 \\ \phi_{41}(\eta_1^2+\eta_2^2) + \phi_{42}\zeta_1^2 + \phi_{43}\zeta_2^2 + \phi_{44}\zeta_1\zeta_2 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix},\ g_1(\eta) = \begin{bmatrix} g_{11}\eta_1 + g_{12}\eta_2 \\ -g_{12}\eta_1 + g_{11}\eta_2 \end{bmatrix},$$
$$g_2(\zeta) = \begin{bmatrix} g_{31}\zeta_1 + g_{32}\zeta_2 \\ g_{41}\zeta_2 + g_{42}\zeta_2 \end{bmatrix}$$

The open forms of the parameters above can be found in [5]. The input shall be fixed to be $u = K\rho^2$ from this point on[1].

If one represents the above in polar coordinates, i.e. $\rho = \sqrt{\eta_1^2 + \eta_2^2}$ and $\theta = \arctan(\eta_2/\eta_1)$ one gets

$$\begin{aligned} \dot{\rho} &= \left(\sigma + \phi_{1,1}\zeta_1 + \phi_{1,2}\zeta_2 + g_{1,1}K\rho^2\right)\rho \\ \dot{\theta} &= \omega - \phi_{2,1}\zeta_1 - \phi_{2,2}\zeta_2 - g_{1,2}K\rho^2 \\ \dot{\zeta}_1 &= -\lambda_1\zeta_1 + \phi_{3,4}\zeta_1\zeta_2 + \phi_{3,2}{\zeta_1}^2 + \phi_{3,3}{\zeta_2}^2 + \phi_{3,1}\rho^2 \\ &\quad + \left(b_{2,1} + g_{3,1}\zeta_1 + g_{3,2}\zeta_2\right)K\rho^2 \\ \dot{\zeta}_2 &= -\lambda_2\zeta_2 + \phi_{4,4}\zeta_1\zeta_2 + \phi_{4,2}{\zeta_1}^2 + \phi_{4,3}{\zeta_2}^2 + \phi_{4,1}\rho^2 \\ &\quad + \left(b_{2,2} + g_{4,1}\zeta_1 + g_{4,2}\zeta_2\right)K\rho^2 \end{aligned} \tag{12}$$

[1] A few comments on this input selection: First of all, note that it would not have been possible to use $u = K\rho$ instead, since this is not smooth at $\eta = 0$. Second, it is also possible to obtain the above form using a phase dependent control, i.e. one that includes θ: It is possible to show that, for a control of the form $u = \frac{\pi}{2}K\rho^2\sin(\frac{1}{2}\theta)$, the equations for the averaged system will be identical to those in (11).

From (12), treating σ as a state with trivial dynamics, one can write

$$\begin{bmatrix} \dot{\sigma} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma \\ \rho \end{bmatrix} + \begin{bmatrix} \varphi_{11}(\sigma,\rho,\zeta) \\ \varphi_{12}(\sigma,\rho,\zeta) \end{bmatrix}$$
$$\begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} \varphi_{21}(\sigma,\rho,\zeta) \\ \varphi_{22}(\sigma,\rho,\zeta) \end{bmatrix} \tag{13}$$

where $\varphi_1 = [\varphi_{11}\ \varphi_{12}]^T$ and $\varphi_2 = [\varphi_{21}\ \varphi_{22}]^T$ encapsulate the nonlinear terms in (12). Clearly $\varphi_1(0,0,0) = \varphi_2(0,0,0) = 0$. Hence for system (13) there exists a center manifold, $\zeta = \bar{\zeta}(\sigma,\rho)$, where $\bar{\zeta} = [\bar{\zeta}_1\ \bar{\zeta}_2]^T$, satisfying

$$\frac{\partial\bar{\zeta}_i}{\partial\rho}\varphi_{12}(\sigma,\rho,\bar{\zeta}(\sigma,\rho)) = -\lambda_i\bar{\zeta}_i(\sigma,\rho) + \varphi_{2i}(\sigma,\rho,\bar{\zeta}_i(\sigma,\rho)) \tag{14}$$

for $i = 1, 2$, to which the dynamics of the system (13) will be locally attracted [14].

The differential equations in (14) are too complicated to be solved directly, so one needs to look for an approximation of the form

$$\begin{aligned} \bar{\zeta}_i(\rho,\sigma) &= c_{i,0}(\sigma) + c_{i,1}(\sigma)\rho + c_{i,2}(\sigma)\rho^2 + c_{i,3}(\sigma)\rho^3 \\ &\quad + c_{i,4}(\sigma)\rho^4 + \mathcal{O}(\rho^5) \end{aligned} \tag{15}$$

subject to conditions $c_{i,0}(0) = 0$, $c_{i,1}(0) = 0$, so as to satisfy the center manifold requirements $\bar{\zeta}_i(0,0) = 0$, $D\bar{\zeta}_i(0,0) = 0$, for $i = 1, 2$. Substituting (15) into (14) and solving for the coefficients gives:

$$\begin{aligned} c_{1,0} &= 0\ ,\ c_{2,0} = 0,\ c_{1,1} = 0\ ,\ c_{2,1} = 0 \\ c_{1,2} &= \frac{2\,\sigma\,Kb_{2,1} + 2\,\sigma\,\phi_{3,1} + \lambda_2 Kb_{2,1} + \lambda_2\phi_{3,1}}{4\,\sigma^2 + 2\,\sigma\,\lambda_2 + 2\,\lambda_1\sigma + \lambda_1\lambda_2} \\ c_{2,2} &= \frac{2\,\sigma\,Kb_{2,2} + 2\,\sigma\,\phi_{4,1} + \lambda_1 Kb_{2,2} + \lambda_1\phi_{4,1}}{4\,\sigma^2 + 2\,\sigma\,\lambda_2 + 2\,\lambda_1\sigma + \lambda_1\lambda_2} \end{aligned}$$

$$\begin{aligned} c_{1,3} &= 0\ ,\ c_{2,3} = 0 \\ c_{1,4} &= [\lambda_2 c_{2,2}g_{3,2}K + \lambda_2 c_{1,2}g_{3,1}K + \lambda_2 c_{2,2}c_{1,2}\phi_{3,4} \\ &\quad -2\,\lambda_2 c_{1,2}\phi_{1,2}c_{2,2} - 2\,\lambda_2 c_{1,2}g_{1,1}K \\ &\quad +4\,\sigma\,c_{2,2}c_{1,2}\phi_{3,4} + 4\,\sigma\,c_{2,2}g_{3,2}K + 4\,\sigma\,c_{1,2}g_{3,1}K \\ &\quad -8\,\sigma\,c_{1,2}\phi_{1,2}c_{2,2} - 8\,\sigma\,c_{1,2}g_{1,1}K - 8\,\sigma\,\phi_{1,1}{c_{1,2}}^2 \\ &\quad +4\,\sigma\,{c_{1,2}}^2\phi_{3,2} + 4\,\sigma\,{c_{2,2}}^2\phi_{3,3} - 2\,\lambda_2\phi_{1,1}{c_{1,2}}^2 \\ &\quad +\lambda_2{c_{1,2}}^2\phi_{3,2} + \lambda_2{c_{2,2}}^2\phi_{3,3}] \\ &\quad /\,[4\,\sigma\,\lambda_2 + 4\,\lambda_1\sigma + \lambda_1\lambda_2 + 16\,\sigma^2] \\ c_{2,4} &= [\lambda_1 c_{2,2}c_{1,2}\phi_{4,4} - 2\,\lambda_1 c_{2,2}g_{1,1}K - 2\,\lambda_1 c_{2,2}\phi_{1,1}c_{1,2} \\ &\quad +4\,\sigma\,c_{1,2}g_{4,1}K + 4\,\sigma\,c_{2,2}g_{4,2}K + 4\,\sigma\,c_{2,2}c_{1,2}\phi_{4,4} \\ &\quad -8\,\sigma\,c_{2,2}g_{1,1}K - 8\,\sigma\,c_{2,2}\phi_{1,1}c_{1,2} + \lambda_1 c_{1,2}g_{4,1}K \\ &\quad +\lambda_1 c_{2,2}g_{4,2}K + 4\,\sigma\,{c_{2,2}}^2\phi_{4,3} + 4\,\sigma\,{c_{1,2}}^2\phi_{4,2} \\ &\quad +\lambda_1{c_{2,2}}^2\phi_{4,3} + \lambda_1{c_{1,2}}^2\phi_{4,2} - 8\,\sigma\,\phi_{1,2}{c_{2,2}}^2 \\ &\quad -2\,\lambda_1\phi_{1,2}{c_{2,2}}^2] \\ &\quad /\,[4\,\sigma\,\lambda_2 + 4\,\lambda_1\sigma + \lambda_1\lambda_2 + 16\,\sigma^2]\ . \end{aligned}$$

If one fully expands, simplifies and collects terms, a structure of following form is obtained

$$\begin{aligned} c_{1,2} &= \mu_{1,2,1} + \mu_{1,2,2} K \\ c_{2,2} &= \mu_{2,2,1} + \mu_{2,2,2} K \\ c_{1,4} &= \mu_{1,4,1} + \mu_{1,4,2} K + \mu_{1,4,3} K^2 \\ c_{2,4} &= \mu_{2,4,1} + \mu_{2,4,2} K + \mu_{2,4,3} K^2 \ . \end{aligned} \quad (16)$$

Substituting equations (16) into (15) and then substituting this into (12) and collecting terms gives

$$\dot{\rho} = (\sigma + d_1\rho^2 + d_2\rho^4)\rho \quad (17)$$

where

$$\begin{aligned} d_1 &= \phi_{1,2}\left(\mu_{2,2,1} + \mu_{2,2,2}K\right) + \phi_{1,1}\left(\mu_{1,2,1} + \mu_{1,2,2}K\right) \\ d_2 &= \phi_{1,1}\left(\mu_{1,4,1} + K\mu_{1,4,2} + K^2\mu_{1,4,3}\right) \\ &\quad + \phi_{1,2}\left(\mu_{2,4,1} + K\mu_{2,4,2} + K^2\mu_{2,4,3}\right) \end{aligned} \quad (18)$$

In (17) the positive roots $\{\rho_1, \rho_2\}$ of the polynomial $\sigma + d_1\rho^2 + d_2\rho^4$ are

$$\{\rho_1, \rho_2\} = \sqrt{\frac{-d_1 \pm \sqrt{d_1^2 - 4d_2\sigma}}{2d_2}} \quad (19)$$

The roots of (19) will now be analyzed based on the signs of d_1 and d_2: First recall that $\sigma > 0$. Next note that it is required to have $d_1^2 - 4\sigma d_2 > 0$ to have any real roots. Assuming this is the case from now on:

If it is the case that $d_2 > 0$, in order to have real roots for (19) then it is required that $-d_1 \pm \sqrt{d_1^2 - 4\sigma d_2} > 0$. If $d_1 > 0$, then $-d_1 - \sqrt{d_1^2 - 4\sigma d_2} > 0$ is impossible and to have $-d_1 + \sqrt{d_1^2 - 4\sigma d_2} > 0$ it is required that $d_2 < 0$ which is also not true for this particular case; hence $d_2 < 0,\ d_1 > 0$ case gives no real solutions. If $d_1 < 0$, then $-d_1 + \sqrt{d_1^2 - 4\sigma d_2} > 0$ is clearly true, and $-d_1 - \sqrt{d_1^2 - 4\sigma d_2} > 0$ is also true since $d_2 > 0$ for this case. Hence for the $d_2 < 0,\ d_1 < 0$ case there are two real solutions.

If it is the case that $d_2 < 0$, in order to have real roots for (19) it is required that $-d_1 \pm \sqrt{d_1^2 - 4\sigma d_2} < 0$. If $d_1 > 0$, then $-d_1 + \sqrt{d_1^2 - 4\sigma d_2} < 0$ cannot hold as $d_2 > 0$. However $-d_1 - \sqrt{d_1^2 - 4\sigma d_2} < 0$ is true so for this case there is a single real solution. for the $d_2 < 0,\ d_1 > 0$ case. If $d_1 < 0$, then $-d_1 + \sqrt{d_1^2 - 4\sigma d_2} < 0$ is false but $-d_1 - \sqrt{d_1^2 - 4\sigma d_2} < 0$ is true for this case, so there is also one real solution for the $d_2 < 0,\ d_1 < 0$. Combining the two cases the conclusion is that there exists a single solution for $d_2 < 0$.

V. EXAMPLE

The Galerkin model is widely used in flow control as a reduced order model describing the dynamics of the flow. An interesting case is the control of the air flow over a cavity using an synthetic-like jet actuator, which is typically an acoustic actuator (see Figure 1).

In deriving a reduced order model for the cavity flow process, one starts with the incompressible Navier-Stokes equation that describes the dynamics of the cavity flow

$$\partial_t u + \nabla \cdot (u\ u) = -\nabla p + \frac{1}{Re}\Delta u \quad (20)$$

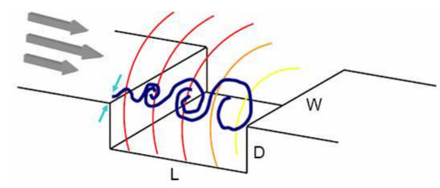


Fig. 1. Control of air flow over a cavity using an acoustic actuator.

subject to the some boundary conditions in which the system input is embedded. Here $u(x,t)$ is the velocity field, p is the pressure and Re is the Reynolds number. One then obtains a set of POD modes for the system; let these modes be modes $\{u_i(x)\}_{i=0}^N$. These POD modes are orthonormal i.e. $(u_i, u_j)_\Omega = \delta_{ij}$ where Ω is the spatial flow domain, and the inner product is defined as $(u,v) := \int_\Omega u \cdot v\, dV$.

Projecting the velocity vector u onto these modes one obtains the POD expansion as:

$$u(x,t) \approx u^{[N]} = u_0(x) + \sum_{i=1}^{N} a_i(t) u_i(x) \quad (21)$$

where the coefficients $a_i(t)$ capture time dependence.

The next step is the Galerkin Projection (GP) where (21) is substituted into (20) to obtain the dynamics in terms of the time coefficients $\{a_i(t)\}_{i=0}^N$, followed by a shift by the equilibrium point a_s, i.e. $\tilde{a} = a - a_s$, to obtain

$$\dot{\tilde{a}}_i = \frac{1}{Re}\sum_{j=1}^{N} \tilde{l}_{ij}\tilde{a}_j + \sum_{j,k=1}^{N} \tilde{q}_{ijk}\tilde{a}_j\tilde{a}_k + \left(\tilde{r}_i + \sum_{j=0}^{N} \tilde{s}_{ij}\tilde{a}_j\right) u$$

where l_{ij}, q_{ijk}, r_i and s_{ij} are the Galerkin system coefficients. Comparing this to (1) one sees that the system is now of the form that was analyzed in this paper. Hence for a control of the form $u = K\rho^2$, the corresponding reduced system is of the form (17). For numerical simulation, parameter values from the cavity flow experimental setup descried in [15] will be used. Using these values to compute d_1 and d_2 from (18)

$$\begin{aligned} d_1 &= -5.0579\ 10^{-4}K - 0.0876 \\ d_2 &= -7.4279\ 10^{-8}K^2 - 1.0121\ 10^{-4} - 0.0149 \end{aligned}$$

and from the above equations, the discriminant can be computed as

$$d_1^2 - 4\sigma d_2 = 3.1053\ 10^{-7}K^2 + 1.6316\ 10^{-4}K + 0.0187$$

Carrying out an analysis similar to that in the previous section, one concludes that, there will be no positive real solutions for $K \in (-1194.3, -168.3783)$, one positive real solution for $K \in (-\infty, -1194.3) \cup (-168.2720, \infty)$, and two positive real solutions for $K \in (-168.3783, -168.2720)$.

Figure 2 shows the change of the oscillation amplitude ρ^* versus the controller gain K, for control $u = K\rho^2$, in the range $K \in [-1000, 1000]$. The graph is computed based on the center manifold analysis above. In this range

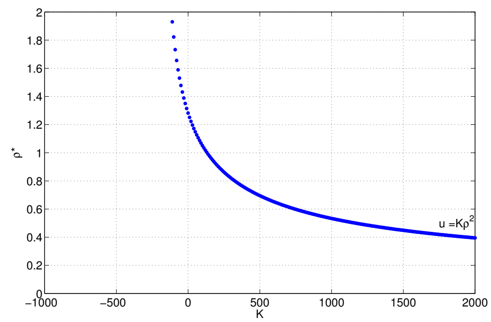
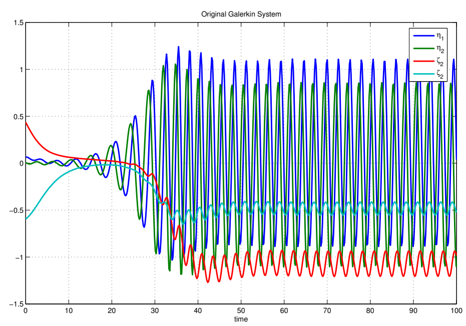
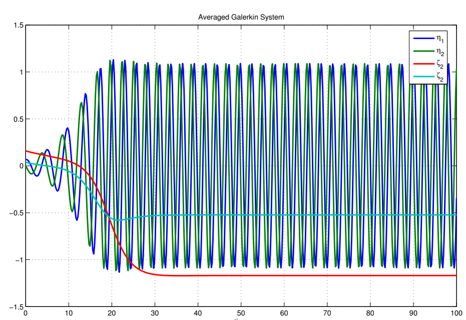


Fig. 2. Steady state oscillation amplitude ρ^* vs. controller gain K



(a) Original



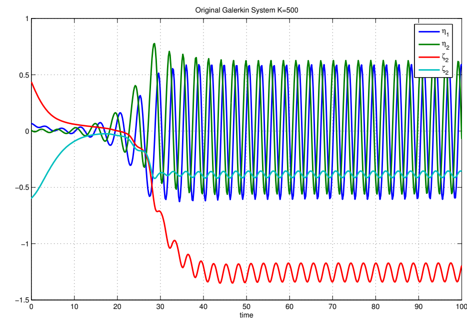
(b) Averaged

Fig. 3. The original and averaged Galerkin systems with no control.

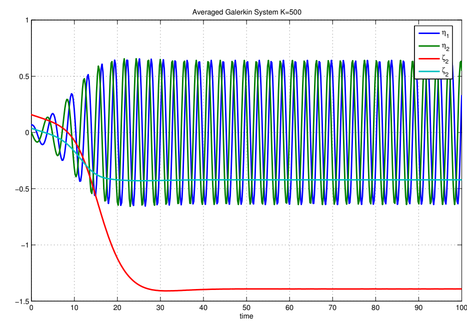
for K, one can observe that the positive values of K decrease ρ^* whereas positive values increase it up until around $K \approx -200$ after which the system does not have a stable oscillation.

VI. SIMULATION RESULTS

To illustrate the validity of the analysis done in the preceding sections and to compare how well the averaged reduced system approximates the original Galerkin system, a number of MATLAB simulations were performed on the original (finite-dimensional) Galerkin system as well as the averaged Galerkin system. As mentioned in the previous section we used the parameters from the cavity flow experimental setup in [15].

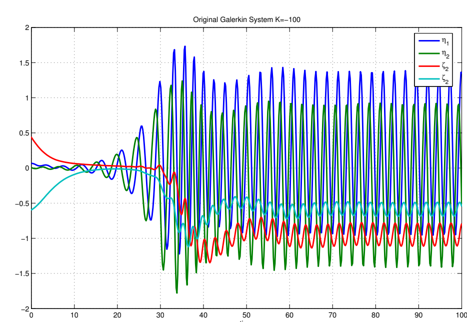


(a) Original

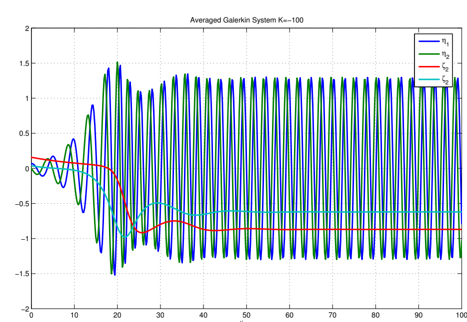


(b) Averaged

Fig. 4. The original and averaged Galerkin systems with control $u = K\rho^2$ where $K = 500$.

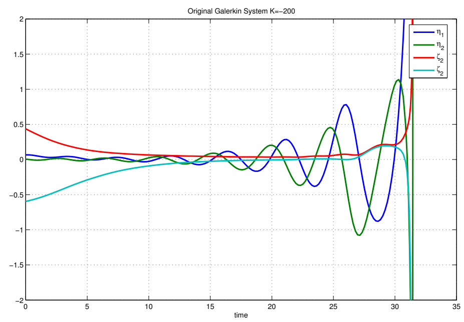


(a) Original

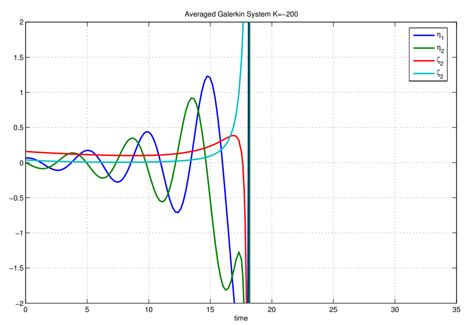


(b) Averaged

Fig. 5. The original and averaged Galerkin systems with control $u = K\rho^2$ where $K = -100$.



(a) Original



(b) Averaged

Fig. 6. The original and averaged Galerkin systems with control $u = K\rho^2$ where $K = -200$.

Figure 3 shows the simulation results for the original and averaged system in the open loop. The system eventually sustains an oscillation of amplitude around 1, which is consistent with what is predicted by the center manifold analysis in figure 2.

Figures 4 shows the simulation results for the original and averaged system for control $u = K\rho^2$ with $K = 500$. One observes that the system still has a stable oscillation but the amplitude of oscillation is reduced to around 0.6-0.7, which is also consistent with the center manifold analysis in figure 2.

Figures 5 shows the simulation results for the original and averaged system for control $u = K\rho^2$ with $K = -100$. One observes that the oscillation amplitude now increased to around 1.5. This sort of behavior was also predicted by the center manifold analysis, as can bee seen in figure 2.

VII. CONCLUSIONS AND FUTURE WORK

In this paper, a nonlinear control system whose dynamics is described by a Galerkin model was studied. An analysis on the effect of linear control was performed using modal forms and center manifold theory. To analyze the effect of nonlinear control a simplification of the system model was performed using a time varying periodic change of coordinates, a time scaling, and averaging. It was shown that for certain types of control, this procedure yields a much simpler system with much more apparent structure than the original. The averaged system was further simplified using center manifold theory, after which it was possible to obtain conditions governing the number and stability type of the limit cycles of the closed loop system, and to derive analytical expressions for the amplitude of oscillation. The results obtained in the study were tested and verified using the cavity flow control problem as an example, whose dynamics can be described by a Galerkin model of the type considered here.

Future directions include: expanding the results to other families of control laws, observer design for the Galerkin model, and verifying the results on an actual cavity flow experimental setup.

VIII. ACKNOWLEDGMENTS

The support of AFRL/VA and AFOSR though the CCCS is gratefully acknowledged. The authors would also like to thank Dr. Bernd Noack and Dr. Gilead Tadmor for fruitful and insightful discussions.

REFERENCES

[1] S. Behtash and S. Sastry. Stabilization of nonlinear systems with uncontrollable linearization. *IEEE Transactions on Automatic Control*, 33(6):585–90, 1988.

[2] D. Cheng and C. Martin. Stabilization of nonlinear systems via designed center manifold. *IEEE Transactions on Automatic Control*, 46(9):1372 – 1383, Sept 2001.

[3] N. H. El-Farra, A. Armaou, and P. D. Christofides. Analysis and control of parabolic PDE systems with input constraints. *Automatica*, 39(4):715–25, 2003.

[4] K. Fitzpatrick, Y. Feng, R. Lind, A. J. Kurdila, and D. W. Mikolaitis. Flow control in a driven cavity incorporating excitation phase differential. *Journal Of Guidance, Control, And Dynamics*, 28(1):63–70, January 2005.

[5] C. Kasnakoglu. Analysis and nonlinear control of Galerkin models using averaging and center manifold theory (long version). Technical report, OSU Deptartment of ECE, 2006.

[6] H. K. Khalil. *Nonlinear Systems*. Prentice Hall,, Upper Saddle River, NJ :, c1996.

[7] W. Lin and C.I. Byrnes. Design of discrete-time nonlinear control systems via smooth feedback. *IEEE Transactions on Automatic Control*, 39(11):2340 – 2346, Nov 1994.

[8] B.R. Noack, G. Tadmor, and M. Morzyński. Actuation models and dissipative control in empirical Galerkin models of fluid flows. In *The 2004 American Control Conference*, pages 0001–0006, Boston, MA, U.S.A., 2004.

[9] B.R. Noack, G. Tadmor, and M. Morzynski. Low-dimensional models for feedback flow control. Part I: Empirical Galerkin models. In *2nd AIAA Flow Control Conference*, Portland, Oregon, U.S.A., June 28 – July 1, 2004, 2004. AIAA Paper 2004-2408.

[10] O. K. Rediniotis, Jeonghwan Ko, and A. J. Kurdila. Reduced order nonlinear Navier-Stokes models for synthetic jets. *Transactions of the ASME. Journal of Fluids Engineering*, 124(2):433–443, 2002.

[11] C. W. Rowley and V. Juttijudata. Model-based control and estimation of cavity flow oscillations. In *CDC-ECC '05 44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference.*, pages 512 – 517, Dec 2005.

[12] S. N. Singh, J. H. Myatt, G. A. Addington, S. Banda, and J. K. Hall. Optimal feedback control of vortex shedding using proper orthogonal decomposition models. *Transactions of the ASME. Journal of Fluids Engineering*, 123(3):612–618, 2001.

[13] G. Tadmor, B.R. Noack, M. Morzynski, and S. Siegel. Low-dimensional models for feedback flow control. Part II: Controller design and dynamic estimation. In *2nd AIAA Flow Control Conference*, Portland, Oregon, U.S.A., June 28 – July 1, 2004, 2004. AIAA Paper 2004-2409.

[14] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. Springer-Verlag,, New York :, c1990.

[15] X. Yuan, E. Caraballo, M. Debiasi, J. Little, A. Serrani, H. Özbay, and M. Samimy. Experimental results and bifurcation analysis on scaled feedback control for subsonic cavity flows. In *Proceedings of the 2006 Mediterranean Control Conference*, Ancona, Italy, 2006.