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# Control of Nonlinear Systems Represented by Galerkin Models Using Adaptation-based Linear Parameter-varying Models

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**Abstract:** This paper studies the control of nonlinear Galerkin systems, which are an important class of nonlinear systems that arise in reduced-order modeling of infinite-dimensional systems. A novel approach is proposed in which a linear parameter-varying (LPV) model representing the Galerkin model is built, where the parameter variation is dictated by a specially designed adaptation scheme. The controller design is then carried out on the simpler LPV model, instead of dealing directly with the complicated nonlinear Galerkin system. An automatically scheduled H-infinity controller is designed using the LPV model, and it is proven that this controller will indeed achieve the desired stabilization when applied to the nonlinear Galerkin model. The approach is illustrated with an example on cavity flow control, where the design is seen to produce satisfactory results in suppressing unwanted oscillations.

**Keywords:** Adaptation, cavity flow, flow control, Galerkin systems, H-infinity control, linear parameter varying (LPV) systems, self scheduling.

## 1. INTRODUCTION

Systems in many areas and applications are described by dynamics which are quadratic with respect to the state and the input, and bilinear with respect to cross terms in the state and input. A system model of this type is the Galerkin model, which is an important class of nonlinear models that arise in reduced-order modeling of infinite-dimensional systems. One can find myriad studies and research work on Galerkin models in literature; to list a few, [1] presented an approach to simulate wave-structure interaction dynamics and constructed a low-dimensional Galerkin model incorporating up to 12 modes and time-dependent boundary conditions. [2] analyzed the chaotic behavior of a Galerkin model of the Kolmogorov fluid motion equations. [3] presented a finite element model for precipitate nucleation and growth during the quench phase of aluminium alloy manufacturing processes, developing a discontinuous Galerkin model for steady advection-diffusion problems to predict the thermal response in a continuous quench process. [4] analyzed a slightly modified Galerkin model which uses the conservative momentum equations, for surface water flow. [5] investigated nonlinear three-dimensional convectionunder gravity modulation using a minimum Galerkin model which describes the competition between several regular convection patterns. [6] presented a bifurcation scenario using a Galerkin model to analyze the nonlinear interactions between rolls and waves and find that they maintain the system in the vicinity of the oscillatory instability onset, thus preventing the blow-up of the growing nonlinear roll solution. [7] studied the conditions for $\mathscr{H}_\infty$ convergence of the transfer functions of finite-dimensional Galerkin approximations for linear distributed-parameter flexible mechanical systems. [8] used selective decay and dynamic alignment relaxation theories to interpret the time asymptotic behavior of a Galerkin model of three-dimensional (3-D) magnetohydrodynamics. [9] investigated currents of heat, concentration, and mass in binary fluid layers heated from below using nonlinear analytical solutions of a simple Galerkin model for impermeable horizontal boundary conditions. [10] presented a few-mode Galerkin model for convection in binary fluid layers subject to impermeable horizontal boundary conditions at positive separation ratios. [11] studied physicalphenomena fundamental to rotating baroclinically driven flows, with reference to results of numerical simulation of rotating annulus flows, using a modified Galerkin model.

One of the fields in which Galerkin models are widely used is aerodynamic flow control, which has been an initial motivation for the study in this paper. In flow control problems, one usually obtains an orthonormal set of basis functions for flow variables, e.g., using proper orthogonal decomposition (POD), and then projects the governing equations onto a finite number of modes using Galerkin projection (GP). This produces a Galerkin model that approximates the original system of nonlinear partial differential equations (PDEs). An approach of this sort has been used, among others, in feedback control of cylinder wakes [12-17], control of cavity flow [18-24], and optimal control of vortex shedding [25,26].

While Galerkin models provide a reduced order approximation for many infinite dimensional processes

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represented by PDEs, their nonlinear nature demands the use of specialized and complicated nonlinear control theoretic methods for analysis and control design. One possibility towards further simplification is to use linearization; however, this will limit the analysis and control design to a single operating condition, which is unacceptable for many problems. In this paper we propose a novel and less restrictive approach for the analysis and control of nonlinear Galerkin models, using techniques based on adaptation [27-31]. We demonstrate that, instead of dealing directly with the nonlinear Galerkin model, one can build a Linear Parameter Varying (LPV) model representing the Galerkin model. LPV models are those which have a linear structure, but some of the system parameters vary with time. For our approach, the parameter variation of the LPV model is controlled by a specially designed adaptation scheme. From control theoretic perspective, handling an LPV system is preferable to dealing with the nonlinear Galerkin model, since numerous techniques have been devised for the control of LPV systems, including self-scheduled $\mathscr{H}_\infty$ control approaches [32-34]. It will be shown that with the adaptation mechanism designed using the steps outlined, one can carry out the controller design on the LPV model, and then apply this same controller to the nonlinear Galerkin system with successful results. The ideas described in the paper are exemplified by a flow control case study, namely, the suppression of unwanted oscillations caused by air flow over a cavity.

## 2. PROBLEM DESCRIPTION

In this section we provide the problem description and the main goals of the paper.

**Definition 1** (Galerkin model)**:** Consider a dynamical system model of the following form

$$\begin{aligned}\dot{a}_i = \sum_{k=1}^{n} L_{ik} a_k + \sum_{k=1}^{n} L_{\text{in},ik} u_k + \sum_{j=1}^{n}\sum_{k=1}^{n} Q_{ijk} a_k a_j \\ + \sum_{j=1}^{m}\sum_{k=1}^{n} Q_{\text{ain},ijk} a_k u_j + \sum_{j=1}^{m}\sum_{k=1}^{m} Q_{\text{in},ijk} u_k u_j\end{aligned} \tag{1}$$

for $i = 1,\ldots,n$ where $a = \{a_i\}_{i=1}^n \in \mathbb{R}^n$ is the state vector, $u = \{u_i\}_{i=1}^m \in \mathbb{R}^m$ is the control input, and $n, m \in \mathbb{N}$. We refer to system models of the form (1) as *Galerkin models*, which constitute the main interest of the work presented in the paper. System (1) can also be expressed in compact form as

$$\dot{a} = La + L_{in}u + Q(a,a) + Q_{ain}(a,u) + Q_{in}(u,u), \tag{2}$$

where

$$L = \{L_{ij}\}_{i,j=1}^{n} \in \mathbb{R}^{n\times n}, \quad L_{\text{in}} = \{L_{\text{in},ij}\}_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m},$$

$$Q(a,a) = \{a^T Q_i a\}_{i=1}^{n} \in \mathbb{R}^n, \quad Q_i = \{Q_{ijk}\}_{j,k=1}^{n} \in \mathbb{R}^{n\times n},$$

$$Q_{ain}(a,u) = \{a^T Q_{ain,i} u\}_{i=1}^{n} \in \mathbb{R}^n,$$

$$Q_{ain,i} = \{Q_{ain,ijk}\}_{j,k=1}^{n,m} \in \mathbb{R}^{n\times m},$$

$$Q_{\text{in}}(u,u) = \{u^T Q_{in,i} u\}_{i=1}^{n} \in \mathbb{R}^n,$$

$$Q_{in,i} = \{Q_{in,ijk}\}_{j,k=1}^{n} \in \mathbb{R}^{m\times m}.$$

With the definition of a Galerkin model given above, the first goal is to obtain a linear parameter varying (LPV) system

$$\dot{\hat{a}}(t) = \hat{L}((t))\hat{a}(t) + \hat{L}_{in}((t))u(t) + \hat{L}_{err}((t))\big(\hat{a}(t) - a(t)\big), \tag{3}$$

which closely represents the system in (2); here denotes the time-varying parameter vector. In other words, if $e := \hat{a} - a$, then $e$ should remain bounded and small as $t \to \infty$. The second goal is to design a controller for this system that achieves stabilization of the system, as well as keeping the effect of the disturbance caused by the error $e$ within reasonable limits. Section 3 will be concerned with the first goal, whereas Section 4 will deal with the second.

## 3. DESIGN OF A LINEAR PARAMETER VARYING MODEL APPROXIMATING THE GALERKIN SYSTEM THROUGH ADAPTATION

The task considered in this section is the design of an LPV model of the form (3) which will approximate the Galerkin system (2). Note that while the system in (1) is nonlinear in its state $a$ and its input $u$, it is linear in its parameter values contained in $L$, $Q$, $L_{\text{in}}$, $Q_{\text{in}}$ and $Q_{\text{ain}}$ since there are no terms involving multiplication of two parameter values. To write the dynamics in a from where this linear dependance in apparent, let us first build the parameter vector $\theta \in \mathbb{R}^p$ as

$$\theta := \text{col}\big(L(:), L_{\text{in}}(:), Q(:), Q_{\text{in}}(:), Q_{\text{ain}}(:)\big), \tag{4}$$

where $p \in \mathbb{N}$ is the total number of parameters, col stands for column vector, i.e., $\text{col}(x_1, x_2, \ldots, x_n) = [x_1^T\ x_2^T \ldots x_n^T]^T$, and $L(:)$ denotes the column vector formed by stacking all elements of $L$ on top of each other, i.e.,

$$L(:) := \text{col}(L_{11}, L_{21}, \ldots, L_{n-1,n}, L_{nn}). \tag{5}$$

The definitions for $L_{\text{in}}(:)$, $Q(:)$, $Q_{\text{in}}(:)$ and $Q_{\text{ain}}(:)$ follow similarly. One can then define $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^p$ such that

$$\dot{a} = \Phi(a,u)\theta, \tag{6}$$

where $\Phi(a,u)$ is a $n \times p$ matrix with elements $\{\Phi(a,u)_{ij} \mid i = 1,\ldots,n, j = 1,\ldots,p\}$. Here, $\Phi(a,u)_{ij}$ is the element at row $i$ and column $j$, and corresponds to the contribution of the $j$ th parameter in $\theta$ to the $i$ th state of $a$. For instance, from (4) one sees that the second parameter in $\theta$ is the second parameter of $L$, which from (5) is seen to be $L_{21}$. It can also be seen that the second element in the state vector $a=\text{col}\{a_1, a_2, a_3, \dots, a_n\}$ is $a_2$. Looking at (1), if we highlight the portion of the dynamics of $a_2$ where the term $L_{21}$ appears

$$\dot{a}_2 = \dots + L_{21}a_1 + \dots,$$

which implies that $\Phi(a,u)_{22} = a_1$. Other elements of $\Phi(a,u)$ can be constructed similarly so as to write the system in the desired linear parameter form shown in (6).

Once the system model in linear parameter form as in (6) is obtained, the next step is the design of the adaptation law. Note first that the Galerkin system in (6) can be written as

$$\begin{aligned}
&\dot{a} = \Phi(a,u)\theta, \\
&\dot{a} = \Phi_L(a,u)\theta_L + \Phi_N(a,u)\theta_N, \qquad (7)\\
&\dot{a} = La + L_{\text{in}}u + Q(a,a) + Q_{\text{ain}}(a,u) + Q_{\text{in}}(u,u),
\end{aligned}$$

where we have split the linear and nonlinear parts of the Galerkin system as

$$\begin{aligned}
&\Phi_L(a,u)\theta_L := La + L_{\text{in}}u, \\
&\Phi_N(a,u)\theta_N := Q(a,a) + Q_{\text{ain}}(a,u) + Q_{\text{in}}(u,u).
\end{aligned} \qquad (8)$$

The goal is to obtain a linear model of the form

$$\dot{\hat{a}} = \Phi_L(a,u)\hat{\theta}_L - ke = \hat{L}a + \hat{L}_{in}u - ke, \qquad (9)$$

whose parameter vector $\hat{\theta}_L$ will be modified by an adaptation mechanism to match the Galerkin system (7). It should be emphasized that it is not the goal to achieve $\hat{\theta}_L \to \theta_L$; this is in fact undesirable, since it would imply that (9) approximates the behavior of (7) around only the origin $a=0$. We would instead like $\hat{\theta}_L$ to be modified to force the state trajectory of (9) to that of (7). In other words, the goal is to make the error $e=\hat{a}-a$ small, which is governed by the following dynamics[1]

$$\dot{e} = \dot{\hat{a}} - \dot{a} = \Phi_L(a,u)\hat{\theta}_L - ke - \Phi_L(a,u)\theta_L - \Phi_N(a,u)\theta_N. \qquad (10)$$

The adaptation mechanism considered for this purpose is of the following form

$$\dot{\hat{\theta}}_L = -\Phi_L^T(a,u)e - \Upsilon(\tilde{\theta}_L,a,u) - \Psi(e,\tilde{\theta}_L,a,u), \qquad (11)$$

where

$$\tilde{\theta}_L := \hat{\theta}_L - \theta_c, \qquad (12)$$

$$\Upsilon(\tilde{\theta}_L,a,u) := k_r\tilde{\theta}_L^* \,\|(a,u)\|^4 + k_s\tilde{\theta}_L^* \,\|e\|^2, \qquad (13)$$

$$\tilde{\theta}_L^* := \begin{cases} 0, & \tilde{\theta}_L = 0; \\ \tilde{\theta}_L / \|\tilde{\theta}_L\|^2, & \tilde{\theta}_L \neq 0, \end{cases} \qquad (14)$$

$$\Psi(e,\tilde{\theta}_L,a,u) := \begin{cases} 0, & \|\text{col}(e,\tilde{\theta}_L)\| < k_x \|\text{col}(a,u)\|; \\ k_d\tilde{\theta}_L, & \|\text{col}(e,\tilde{\theta}_L)\| \geq k_x \|\text{col}(a,u)\| \end{cases} \qquad (15)$$

and $k, k_r, k_s, k_d, k_x \in \mathbb{R}_+$, $\theta_c \in \mathbb{R}^p$ are constants to be selected as part of the design process. It can be shown that, by using this adaptation mechanism with properly selected values of these constants, the error $e=\hat{a}-a$ can be made to remain bounded and approach zero. This means that the state trajectories of the system (9) will asymptotically approach those of the Galerkin system (2). We postpone the proof of this statement until Theorem 1 in the Section 6; however we note that if this is the case, then the following interpretation can be made: Let us rearrange (9) as

$$\begin{aligned}
&\dot{\hat{a}} = \hat{L}a + \hat{L}_{\text{in}}u - ke, \\
&\dot{\hat{a}} = \hat{L}(\hat{a}-e) + \hat{L}_{\text{in}}u - ke, \qquad (16)\\
&\dot{\hat{a}} = \hat{L}\hat{a} + \hat{L}_{\text{in}}u + \hat{L}_{\text{err}}e,
\end{aligned}$$

where $\hat{L}_{\text{err}} = -(\hat{L}+kI)$. One can then observe that (16) is of the same form as (3). Thus, if the signal $e$ is bounded and small, one can regard system (9) as a linear parameter-varying system that approximates the original system, with the signal $e$ entering as an external disturbance. With this interpretation, one can carry out the control design on (16) as explained in the next section.

**Remark 1:** At this point it will be useful to emphasize that the parameter vector $\theta$ in (4), i.e., the parameters of the nonlinear Galerkin system contained in $L$, $L_{\text{in}}$, $Q$, $Q_{\text{ain}}$ and $Q_{\text{in}}$ are fixed and not time varying. What is time varying are the parameters of the LPV model denoted by $\hat{\theta}_L$. These are the parameters that are modified by adaptation scheme (11) for the purpose of matching the trajectories of the LPV model (16) with those of the Galerkin system (7).

## 4. CONTROL DESIGN

Having obtained an LPV model (16) to approximate the flow process through adaptation, our goal in this section is to design a controller for this model to stabilize the system and also limit the effect of the error term on the dynamics. We would also like to treat adaptation mechanism an exogenous system as far as the controller is concerned and therefore not utilize any knowledge about the governing differential equations of the adaptation law. The controller is only allowed to use the states of the adaptation mechanism, which is the reconstructed state $\hat{a}$, and the parameter vector $\hat{\theta}_L$ and must be able to achieve the desired stabilization for any parameter trajectory $\hat{\theta}_L(t)$ while limiting the effect of the error $e$ on the output. In this section we will design a controller that can achieve these goals based on a robust automatic scheduling method [32-34]. We provide only a brief summary of this method below, and the

[1]The system (9) can also be though of an adaptive pseudo-observer; the prefix *pseudo* is due to the fact that that a real observer reconstructs the states from outputs, which is not the case here.

reader interested in details of thefull theory is referred to [32-34].

Consider the following affine linear parameter dependent plant

$$\begin{aligned} \dot{x} &= A(\theta)x + B_1(\theta)w + B_2 u, \\ z &= C_1(\theta)x + D_{11}(\theta)w + D_{12}u, \\ y &= C_2 x + D_{21}w, \end{aligned} \tag{17}$$

where $x$ is the state, $u$ is the control input, $w$ is the disturbance input, $y$ is the signal available for control, and $z$ is the output to be controlled. The parameter vector $\theta$ is available in in real-time and varies in a *polytope heta* of vertices $\theta_1,...,\theta_p$; i.e., $\theta \in \Theta$ where $\Theta := \mathrm{Co}\{\theta_1,\ldots,\theta_p\} := \left\{\sum_{i=1}^{p} \alpha_i\theta_i : \alpha_i \geq 0, \sum_{i=1}^{p} \alpha_i = 1\right\}$ and Co stands for convex hull. We assume that $(A(\theta), B_2)$ is quadratically stabilizable over $\theta$ and $(A(\theta), C_2)$ is quadratically detectable over $\theta$. The main goal is to design a dynamic controller whose input is $y$ and generates $u$ which stabilizes the system (17) while minimizing the gain from $w$ to $z$. For this purpose a linear parameter dependent controller having the following structure is considered

$$\begin{aligned} \dot{\zeta} &= A_K(\theta)\zeta + B_K(\theta)y, \\ u &= C_K(\theta)\zeta + D_K(\theta)y \end{aligned} \tag{18}$$

using which the feedback structure shown in Fig. 1 is built. The closed loop system can be expressed as

$$\begin{aligned} \dot{x} &= A_{cl}(\theta)x + B_{cl}(\theta)w, \\ z &= C_{cl}(\theta)x + D_{cl}(\theta)w, \end{aligned} \tag{19}$$

where

$$\begin{aligned} A_{cl}(\theta) &= A_0(\theta) + \mathscr{B}\Omega(\theta)\mathscr{C}, \\ B_{cl}(\theta) &= B_0(\theta) + \mathscr{B}\mathscr{D}_{21}, \\ C_{cl}(\theta) &= C_0(\theta) + \mathscr{D}_{12}\Omega(\theta)\mathscr{C}, \\ D_{cl} &= D_{11}(\theta) + \mathscr{D}_{12}\Omega(\theta)\mathscr{D}_{21}, \end{aligned} \tag{20}$$

and

$$\Omega(\theta) = \begin{bmatrix} A_K(\theta) & B_K(\theta) \\ C_K(\theta) & D_K(\theta) \end{bmatrix}, \quad A_0 = \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} B_1(\theta) \\ 0 \end{bmatrix}, \quad C_0 = \left(C_1(\theta), 0\right), \quad \mathscr{B} = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix},$$

$$\mathscr{C} = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}, \quad \mathscr{D}_{12} = \left(0, D_{12}\right), \quad \mathscr{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}.$$

The task is to design the controller matrices $A_K(\theta)$, $B_K(\theta)$, $C_K(\theta)$ and $D_K(\theta)$ so as to stabilize the closed loop system (19) while at the same time achieving $\| z \|_2 < \gamma \| w \|_2$ for some $\gamma \in \mathbb{R}_+$ for all permissable parameter trajectories $\theta(t)$. The last item is important since the parameters will be generated by a separate adaptation system as in (9)-(11), which is treated as an exogenous system for control design purposes. The controller must therefore be able to account for all possible parameter trajectories within certain bounds, and achieve the stabilization and disturbance attenuation goal for all possible cases. The reader interested in the detailed procedure for obtaining the controller matrices is referred to [32]. For the problem at hand, the LPV system which we would like to control is given in (16), where the input to the controller is taken to be $y = a = \hat{a} - e$, and the system output is taken to be the $z = \mathrm{col}(a, u) = \mathrm{col}(\hat{a} - e, u)$. The system is also augmented with a known input filter so as to eliminate the parameter dependency from the input coefficients. This results in the following augmented system

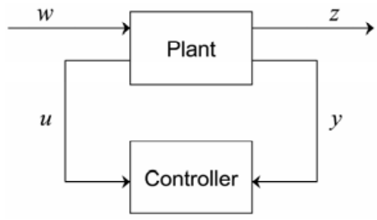


Fig. 1. Feedback structure for linear parameter varying control design.

$$\begin{bmatrix} \dot{\hat{a}} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \hat{L}(\hat{\theta}_L) & \hat{L}_{\mathrm{in}}(\hat{\theta}_L)C_u \\ 0 & A_u \end{bmatrix} \begin{bmatrix} \hat{a} \\ \xi \end{bmatrix} - \begin{bmatrix} \hat{L}(\hat{\theta}_L) + kI \\ 0 \end{bmatrix} e + \begin{bmatrix} 0 \\ B_u \end{bmatrix} v, \tag{21}$$

$$z = \begin{bmatrix} I & 0 \\ 0 & C_u \end{bmatrix} \begin{bmatrix} \hat{a} \\ \xi \end{bmatrix} - \begin{bmatrix} I \\ 0 \end{bmatrix} e, \tag{22}$$

$$y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \hat{a} \\ \xi \end{bmatrix} - e, \tag{23}$$

where $\xi$ is the state vector of the input filter and $A_u$, $B_u$, $C_u$ are its system matrices. Fig. 2 shows a block diagram of the entire system including the nonlinear Galerkin system, the input filter, the adaptation mechanism, andthe controller.

So far we have discussed the main motivation, background information and general path for the method considered in the paper. We are now ready to present the explicit steps of the procedure, and formally analyze its validity.

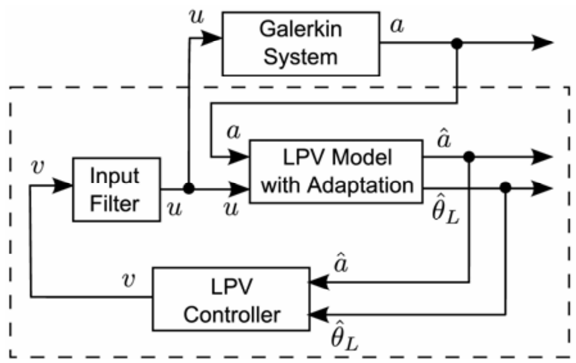


Fig. 2. Block diagram of the entire system.

## 5. STEPS FOR THE PROPOSED APPROACH

**Step 1:** The first step is to set up the LPV model (9) and adaptation mechanism in (11) with the constant $k$ selected to satisfy

$$k > k_6^2 + 1, \tag{24}$$

where

$$k_6 := \max\{\| L \|, \| L_{\text{in}} \|\}. \tag{25}$$

Initially the dissipative terms in (11) are set to zero and the parameter dynamics are allowed to vary freely, i.e., we initially set $\Upsilon = 0$ and $\Psi = 0$.

**Step 2:** The second step is to determine the parameter range $\Theta$ to be used for the LPV model. For this purpose we use typical system identification ideas and excite the system with input signals that are of various amplitudes and frequencies of interest. With these excitations, multiple experiments are conducted where the LPV model (9), the adaptation mechanism (11) and the Galerkin system (7) are run in open-loop, and the resulting parameter trajectories $\hat{\theta}_L(t)$ are recorded. The range $\Theta$ is determined by observing the range in which these parameter trajectories vary. For future reference we express the range $\Theta$ as

$$\Theta = \{\theta \in \mathbb{R}^p : \underline{\theta}_i < \theta(i) < \overline{\theta}_i, \quad i = 1, \ldots, p\}, \tag{26}$$

where $\theta(i) \in \mathbb{R}$ denotes the $i$ th component of $\theta$, and $\overline{\theta}_i$, $\underline{\theta}_i$ are the minimum and maximum values for the $i$ th component of $\theta$. We also define $\Delta\theta_i := \overline{\theta}_i - \underline{\theta}_i$ and

$$\delta_\theta := \min_i \Delta\theta_i. \tag{27}$$

**Step 3:** The third step is to construct the LPV controller, which is itself an LPV system. The controller is of the form (18), and is obtained using a robust self-scheduled control design method [32] as described in Section 4. The bound $\gamma$ should be selected as the smallest possible value for which a feasible solution $X_{cl}$ to the matrix inequalities can be obtained. For future reference we define $\lambda_{\min}$ and $\lambda_{\max}$ as the minimum and maximum eigenvalues of $X_{cl}$.

**Step 4:** The next step the is to incorporate the dissipative terms $\Upsilon$ and $\Psi$ into the adaptation dynamics. The constants in (12)-(15) are selected to satisfy

$$\theta_c := \frac{1}{2}(\overline{\theta}_1 + \underline{\theta}_1, \overline{\theta}_2 + \underline{\theta}_2, \ldots, \overline{\theta}_p + \underline{\theta}_p), \tag{28}$$

$$k_r > k_7^2, \tag{29}$$

$$k_x < \frac{1}{2}\delta_\theta \sqrt{k_{10}^{-1} k_4 k_5^{-1}} k_{\text{IC}}^{-1}, \tag{30}$$

$$k_d > k_x^{-1}, \tag{31}$$

$$k_s > \max\{k_{65}^2 + k_x^{-1}, k_{65}^2 + \gamma^2\}, \tag{32}$$

where

$$k_4 := \min\{\lambda_{\min}, \frac{1}{2}\}, \tag{33}$$

$$k_5 := \max\{\lambda_{\max}, \frac{1}{2}\}, \tag{34}$$

$$k_{65} := \max\{\| L_c \|, \| L_{c,\text{in}} \|\}, \tag{35}$$

$$k_7 := \max\left\{\| Q \| + \frac{\| Q_{\text{ain}} \|}{2}, \| Q_{\text{in}} \| + \frac{\| Q_{\text{ain}} \|}{2}\right\}, \tag{36}$$

$$k_{10} := \max\{1, \| C_u \|\}. \tag{37}$$

In the above, $L_{\text{c}}$ and $L_{\text{c,in}}$ are the matrices whose coefficients form $\theta_c$, i.e.,

$$\theta_c = \left(L_{\text{c}}(:), L_{\text{c,in}}(:)\right) \tag{38}$$

and $k_{\text{IC}}$ is an upper bound on the initial conditions for the entire system, i.e.,

$$\| x_e(0) \| \le k_{\text{IC}}, \tag{39}$$

where $x_e$ includes the states of the Galerkin system, input filter, LPV system, adaptation scheme and the LPV controller. The initial conditions for the LPV model and adaptation scheme are set to

$$(0) = a(0) \tag{40}$$

$$(0) = \theta_c \tag{41}$$

and the rest of the initial conditions can be specified arbitrarily. The addition of the dissipative terms will assure that the parameter trajectory $\hat{\theta}_L(t)$ will remain within $\Theta$ during the closed-loop operation (this statement will be proved in Theorem 1.). This is necessary for the correct operation of the LPV controller.

**Step 5:** The final step is to form the entire system shown in Fig. 2. The dashed lines encapsulate the the elements built in Steps 1-4, namely the input filter, the LPV model and the adaptation scheme, and the LPV controller. These elements collectively form the controller for the nonlinear Galerkin system.

**Remark 2:** Note that the bounds for $\theta$ in Step 2 are obtained prior to controller design. This is inevitable because for the self-scheduled LPV controller design approach utilized [32], the range in which the parameters vary must be known so that the matrix inequalities may be set up at the vertices. Since this paper is a first step towards the idea of using adaptation-based LPV models for the control of Galerkin-type nonlinear systems, we cannot yet formally guarantee that the vertices of the parameter box $\Theta$ resulting from Step 2 will lead to feasible matrix inequalities for all possible cases. However, our experience applying the method to several real-life problems (including the case study in Section 7 of the paper, as well as others) has shown that many cases do indeed result in LPV models with parameter ranges that lead to feasible inequalities for controller design.

Even though it cannot be guaranteed yet that the parameter box $\Theta$ will always yield to solvable inequalities, some heuristic strategies may be tried if no suitable controller can be designed due to conservative bounds from Step 2. For example, instead of a simple box, one can compute the convex hull of the parameter trajectories obtained from the various test signals. This may lead to better conditioned inequalities since the convex hull can cover the parameter trajectories within a smaller volume and than a simple box. One can also utilize an iterative strategy for the design as follows: First, a small range for the parameters (perhaps even constant values) is selected, the matrix inequalities are set up and the controller is designed.Then the parameter trajectories of the closed-loop system are observed to see if they are within the range for which the design was made. If not, the parameter range is enlarged based on the closed-loop parameter trajectories, and the controller design is repeated. Then the closed-loop system is formed with this new controller, the parameter trajectories are recorded and checked against the parameter range. The procedure is repeated until the closed-loop trajectories are in agreement with their design values.

## 6. ANALYSIS OF THE CLOSED-LOOP SYSTEM

In Theorem 1 below, we analyze the closed-loop system to justify the validity of the approach described in the previous section.

**Theorem 1:** Consider the closed-loop system shown in Fig. 2, where the elements inside the dashed box (which collectively form the controller for the nonlinear Galerkin system) are designed as described in Section 5. Then:

1. The trajectories of the LPV system (9), with the parameter vector modified through the adaptation mechanism (11), will converge to the trajectories of the nonlinear Galerkin system (2).
2. The control signal $u$ will asymptotically stabilize the nonlinear Galerkin system (2).
3. The parameter trajectory generated by the adaptation scheme (11) will be contained in the $p$-dimensional box $\Theta$ (26) for all $t \geq 0$.

**Proof:** See Appendix.

The procedure described until this point is illustrated in the next section by a flow control problem that occurs in real-life, namely, suppression of oscillations caused by air flowing over a cavity.

## 7. EXAMPLE: CAVITY FLOW CONTROL

We now consider a physical flow control problem example, namely the suppression of unwanted oscillations generated by the air flow over a shallow cavity. A schematic representation is illustrated in Fig. 3. This is a problem that has captured significant research interest [18,19,22-24], and has been an initial motivation for this study. Air flow over a shallow cavity is characterized by a strong self-sustained resonance produced by a natural feedback mechanism. Shear layer structures impacting the cavity trailing edge scatter acoustic waves that propagate upstream and reach the shear layer receptivity region, where they tune and enhance the development and growth of shear layer structures. The resulting acoustic fluctuations can be very intense and are known to cause, among other problems, store damage and airframe structural fatigue in weapons bay applications. To suppress or reduce the pressure fluctuations inside the cavity, feedback control is applied to the flow by using a synthetic jet actuator, which is typically an acoustic actuator located at the cavity trailing edge [23]. A schematic illustration of a synthetic jet actuator is given in Fig. 4. Synthetic jet actuators are popular for flow control since they synthesize the flow from the surrounding or ambient fluid, and do not require an external source of fluid. A membrane or diaphragm oscillates hundreds of times per second, sucking the surrounding fluid into a chamber and then expelling it.

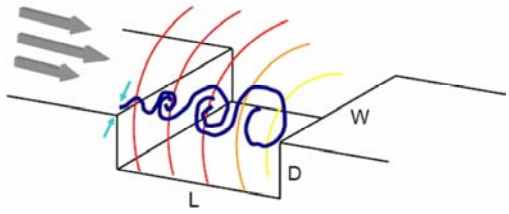


Fig. 3. Control of cavity flow resonance using actuation at the cavity trailing edge. (Figure courtesy of OSU Gas Dynamics and Turbulence Laboratory).

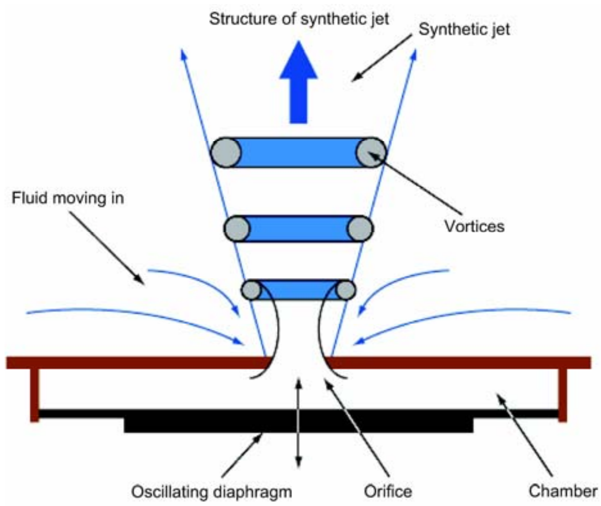


Fig. 4. Schematic diagram of a synthetic jet actuator. (Figure courtesy of electronicdesign.com and Nuventix Inc.).

### 7.1. Obtaining a Galerkin model for cavity flow

The first task is to obtain a Galerkin model for the cavity flow process, which will serve as the system that we wish to control. In deriving the Galerkin model, we start with the Navier-Stokes equations that describe the dynamics of the cavity flow. We treat the flow to be isentropic to simplify the final form of the system. With this treatment, it was shown in [19], that the compressible Navier-Stokes equations can be written as[2]

$$\frac{Du}{Dt}+\frac{1}{M^2}\frac{2}{\kappa-1}\nabla c=\frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u},\qquad \frac{Dc}{Dt}+\frac{\kappa-1}{2}c\,\mathrm{div}\,\boldsymbol{u}=0, \tag{42}$$

where $\boldsymbol{u}(\boldsymbol{x},t)=(u_{\mathrm{s}}(\boldsymbol{x},t),u_{\mathrm{n}}(\boldsymbol{x},t))$ is the flow velocity in the stream-wise and normal direction, $c(\boldsymbol{x},t)$ is the local speed of sound, the operator $D/Dt=\partial/\partial t+\boldsymbol{u}\cdot\nabla$ stands for the material derivative, and $x=(x,y)$ denotes Cartesian coordinates over the spatial domain $\Omega\subset\mathbb{R}^2$. The constants $\kappa$, Re, and $M$ denote respectively ratio of specific heats, Reynolds number, and Mach number. It was also shown in [19] that these equations can be expressed in compact form as

$$\dot{\boldsymbol{q}}=X(\boldsymbol{q}):=\boldsymbol{C}+\boldsymbol{L}(\boldsymbol{q})+\boldsymbol{Q}(\boldsymbol{q},\boldsymbol{q}) \tag{43}$$

defined on the Hilbert space $\mathbb{H}=\mathscr{L}_2(\Omega,\mathbb{R}^2)$ of square-integrable functions on $\Omega$, where $\boldsymbol{q}:=(u_{\mathrm{s}}-u_{\mathrm{s}0},u_{\mathrm{n}}-u_{\mathrm{n}0},c-c_0)\in\mathbb{H}$ is the fluctuations of the flow velocity about the mean value $\boldsymbol{q}_0=(u_{\mathrm{n}0},u_{\mathrm{s}0},c_0)$. In (43), $\boldsymbol{C}$ is a constant operator, $\boldsymbol{L}$ is a linear operator, and $\boldsymbol{Q}(\boldsymbol{q},\boldsymbol{q})$ is quadratic in $\boldsymbol{q}$. It should also be mentioned that the equations above are non-dimensional in the sense that the velocities, coordinates and pressures have all been scaled by appropriate constants. Once the system in written as in (43), it was shown in [35,36] that one can employ Proper Orthonormal Decomposition (POD)/Input Separation (IS) techniques to arrive at an extended expansion of the flow

$$\boldsymbol{q}(\boldsymbol{x},t)=\boldsymbol{q}_0(\boldsymbol{x})+\sum_{i=1}^{n}a_i(t)\phi_i(\boldsymbol{x})+u(t)\psi(\boldsymbol{x}), \tag{44}$$

where $\phi_i$ are the baseline modes, $a_i$ are the POD coefficients, $u$ is the control input and $\psi$ is the actuation mode. The number of baseline modes taken is taken to be four, corresponding to $n=4$ in the POD expansion. This value is a good compromise between the amount of energy captured (about 95% for our case) and the complexity of the reduced-order model. It was also shown in [23,35,36] that substituting (44) into (43), employing Galerkin Projection (GP), shifting by the equilibrium point $a_d$, and transforming into modal form yields a Galerkin model for the cavity flow process of the form

$$\dot{a}=La+Q(a,a)+L_{in}u+Q_{in}(u,u)+Q_{ain}(a,u), \tag{45}$$

where the matrices $L$ and $L_{\mathrm{in}}$ are of the form

$$L=\begin{bmatrix}\sigma & -\omega & 0 & 0\\ \omega & \sigma & 0 & 0\\ 0 & 0 & \lambda_1 & 0\\ 0 & 0 & 0 & \lambda_2\end{bmatrix},\quad L_{in}=\begin{bmatrix}b_1\\ b_2\\ b_3\\ b_4\end{bmatrix} \tag{46}$$

and $\sigma,\omega,\lambda_1,\lambda_2,b_1,b_2,b_3,b_4\in\mathbb{R}$. The system (45) obtained through the procedure described above represents the dynamics of the deviation from the mean flow $\boldsymbol{q}_0$. The control task is to suppress the oscillations causing these deviations, and drive the system back to its mean flow value, i.e., achieve $a\to 0$ as $t\to\infty$. For the numerical simulations below, parameter values from the OSU GDTL cavity flow experimental setup described in [23,35,36] will be used.

7.2. Application of the proposed approach

After the Galerkin model for the flow process is obtained, the next step is to build the LPV model, adaptation scheme and the LPV controller as described in Section 5.

**Step 1:** The first step is to construct the LPV model (9) and the adaptation mechanism (11) without its dissipative terms. We define the parameter vector to be adapted as $\hat{\theta}_L:=\left[\hat{\sigma}\ \hat{\omega}\ \hat{\lambda}_1\ \hat{\lambda}_2\ \hat{b}_1\ \hat{b}_2\ \hat{b}_3\ \hat{b}_4\right]^T$. For future reference let us also denote the individual elements of the state vector of the Galerkin model as $a=[a_1\ a_2\ a_3\ a_4]^T$ and the elements of the reconstructed state vector as $\hat{a}=[\hat{a}_1\ \hat{a}_2\ \hat{a}_3\ \hat{a}_4]^T$. For system (45) in modal form, the matrix $\Phi_L(a,u)$ in (11) can be written as

$$\Phi_L(a,u):=\begin{bmatrix}a_1 & -a_2 & 0 & 0 & u & 0 & 0 & 0\\ a_2 & a_1 & 0 & 0 & 0 & u & 0 & 0\\ 0 & 0 & a_3 & 0 & 0 & 0 & u & 0\\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 & u\end{bmatrix}.$$

The constant $k$ is selected as $k=1000$ to satisfy (24) and also to obtain a reasonable response speed for the adaptation mechanism. The dissipative terms $\Upsilon$ and $\Psi$ are set to zero for this step.

**Step 2:** To determine the range $\Theta$ in which the parameter vector $\hat{\theta}_L$ will vary, the Galerkin system (2), the LPV model (9) and the adaptation system (11) were run together under a high number input signals of various types including ramp functions, sine functions, chirp functions, square waves and white noise, and the values assumed by the parameters under these excitation signals were recorded. As an example we show one of theseinputs in Fig. 5, namely a chirp excitation with its frequency varying between 1 Hz and 0.01 Hz. These frequencies were selected since they roughly represent the top and bottom limits that the actuator is assumed to

[2]These equations have been non-dimensionalized by scaling $u$ by the freestream velocity $U_\infty$, the local speed of sound by the ambient sound speed $c_\infty=(\kappa RT_\infty)^{1/2}$, where $T_\infty$ is the ambient temperature, the cartesian coordinates $x$ by the cavity depth $D$, time by $D/U_\infty$, and pressure by $\bar{\rho}U_\infty^2$, where $\bar{\rho}$ denotes mean density.

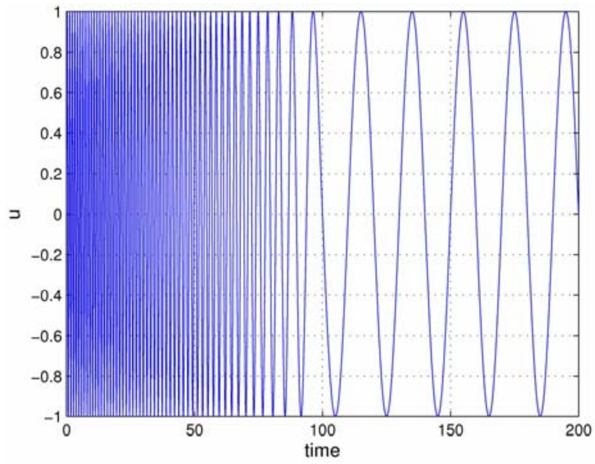


Fig. 5. Chirp signal excitation.

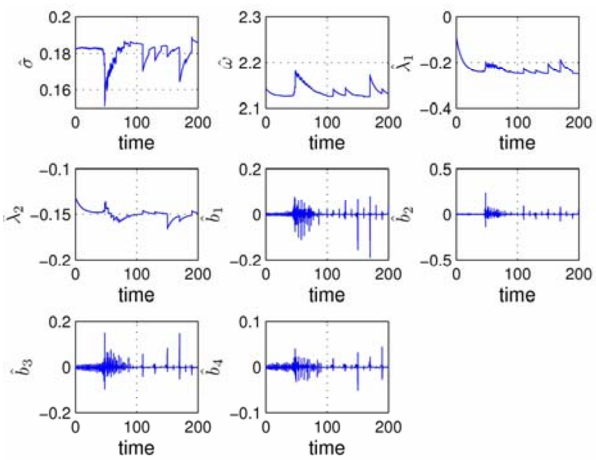


Fig. 6. Parameter vector $\hat{\theta}_L$ under chirp excitation.

be capable of producing.[3] Fig. 6 shows the variation in the parameter values under this chirp excitation. Observing the range in which the parameter values vary for the chirp excitation in the figure, and also for other excitation cases mentioned above (e.g., sine functions, chirp functions, square waves and white noise at various amplitudes and frequencies), the polytope $\Theta$ such that $\hat{\theta}_L \in \Theta$ is chosen to be the 8-dimensional box

$$\begin{aligned}\Theta = \{&\hat{\theta}_L \in \mathbb{R}^8 : -0.0500 < \hat{\sigma} < 0.2500,\\ &1.0000 < \hat{\omega} < 2.8000, -0.3100 < \hat{\lambda}_1 < 0.1700,\\ &-0.3500 < \hat{\lambda}_2 < -0.0400, -0.4400 < \hat{b}_1 < 0.3990,\\ &-0.4300 < \hat{b}_2 < 0.3810, -0.1680 < \hat{b}_3 < 0.2010,\\ &-0.4300 < \hat{b}_2 < 0.3810, -0.1680 < \hat{b}_3 < 0.2010,\\ &-0.0960 < \hat{b}_4 < 0.0730\}.\end{aligned} \tag{47}$$

**Step 3:** The next step in the process is the design of the controller. As explained in Section 4, the approach to control design is to treat the system (16) as an LPV system, where the parameters keep changing with time and are supplied by the adaptation mechanism (11). The parameter values are available in real-time and are to be utilized by the controller, where the control design is based on the self-scheduled $\mathscr{H}_\infty$ control techniques. Forthe cavity flow problem there are eight parameters and the parameter polytope $\Theta$ is a simple box in 8D space given in (47). Once the parameter box $\Theta$ is known, the controller matrices are obtained using results from [32] as described in Section 4. The matrices $A_{cl}$, $B_{cl}$, $C_{cl}$ and $D_{cl}$ given in (20) are

$$A_0 = \begin{bmatrix} \hat{L}(\hat{\theta}_L) & \hat{L}_{in}(\hat{\theta}_L)C_u & 0 \\ 0 & A_u & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} -\hat{L}(\hat{\theta}_L) - kI \\ 0 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & C_u & 0 \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} 0 & 0 \\ 0 & B_u \\ I & 0 \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \end{bmatrix},$$

$$\mathscr{D}_{12} = 0, \; \mathscr{D}_{21} = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

and

$$\hat{L}(\hat{\theta}_L) = \begin{bmatrix} \hat{\sigma} & -\hat{\omega} & 0 & 0 \\ \hat{\omega} & \hat{\sigma} & 0 & 0 \\ 0 & 0 & \hat{\lambda}_1 & 0 \\ 0 & 0 & 0 & \hat{\lambda}_2 \end{bmatrix}, \quad \hat{L}_{in}(\hat{\theta}_L) = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \\ \hat{b}_4 \end{bmatrix}.$$

The matrices $A_u$, $B_u$ and $C_u$ above define an input filter for the system, which for the flow problem at hand, are simply selected so as to yield a first order band-pass Butterworth filter with $f_{\text{low}} = 0.01$ Hz and $f_{\text{high}} = 1$ Hz. This is justified by the fact that the actuation is produced through the motion of an acoustic diaphragm inside the actuator, which can only oscillate between a certain frequency range. Such a simple model will be sufficient for the purposes of this study; the reader interested in further details of the acoustic actuator and more sophisticated models of its dynamics is referred to [37]. The controller matrices $A_{cl}$, $B_{cl}$, $C_{cl}$ and $D_{cl}$ were obtained using the functions of the Robust Control Toolbox in MATLAB® which resulted in a quadratic $\mathscr{H}_\infty$ performance $\gamma = 5.8432$ from $e$ to $z = \mathrm{col}(a,u)$.

**Remark 3:** For the problem studied, the dimension of $\theta$ is eight, which leads to $2^8 = 256$ matrix inequalities to be solved at the vertices of $\Theta$. Although it was possible to find a solution in our case using MATLAB® Robust Control Toolbox, there may be many cases in which these inequalities are not feasible. At this point we cannot yet claim that the LPV systems resulting from the proposed approach will always be amenable to self-scheduled controller design methods [32]; nonetheless, in many cases the proposed approach will indeed result in models to which the method in [32] can be applied and hence the usefulness of the approach in this paper.

**Step 4:** Once the LPV controller is obtained, the

[3] Recall that like all variables, the frequencies are also in the non-dimensionalized scale. The actual frequencies that the actuator can produce are a few thousand times higher.

dissipative terms $\Upsilon$ and $\Psi$ are incorporated into the adaptation dynamics (11), with values in (12)-(15) selected following Step 4 in Section 5 as follows: $k_r = 10$, $k_x = 50$, $k_d = 1$, $k_s = 50$ and $\theta_c = l(0.1000,$ $1.9000, -0.0700, -0.1950, -0.0205, -0.0245, 0.0165,$ $-0.0115)$.

**Step 5:** The final step in the process is to build, implement and test the full system shown in Fig. 2. Figs. 7-10 show the numerical simulation results for this configuration. For test purposes, the control action is set to be $u = 0$ until $t = 100$, so that the system runs in open-loop for this period. The controller is incorporated into the system by closing the loop at $t = 100$. It can be seen from Fig. 8 that the LPV controller is successful in achieving the desired stabilization of driving $\hat{a} \to 0$. Fig. 9 shows the adaptation error $e = \hat{a} - a$, which seems to remain less that $10^{-3}$. The fact that $\hat{a} \to 0$ also implies that $a \to e$, and since error is very small, this practically means that the $a \to 0$ as well, as confirmed from Fig. 7. Fig. 10 shows the state vector

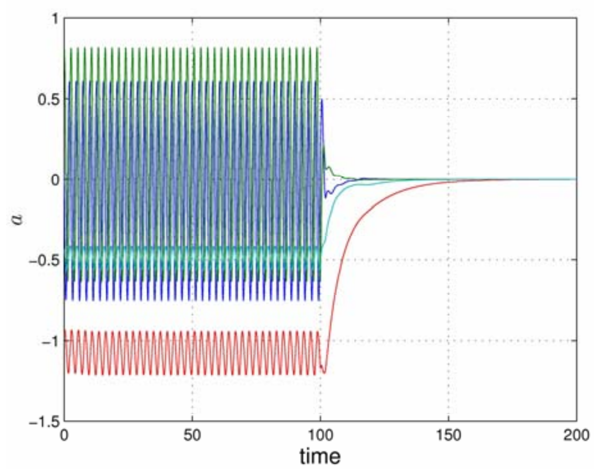

Fig. 7. Flow system states for closed-loop operation with the controller turned on at $t = 100$.

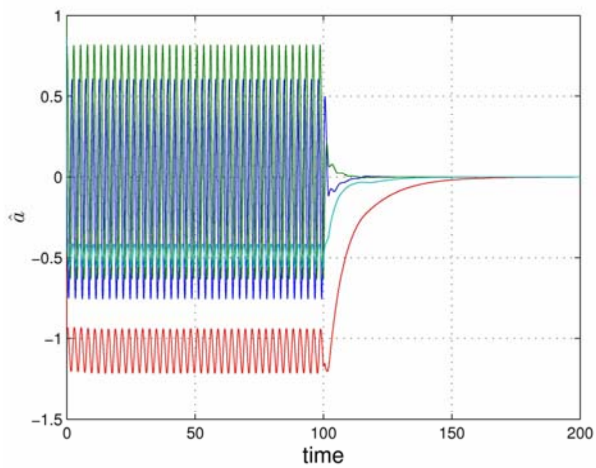

Fig. 8. Reconstructed states $\hat{a}$ for closed-loop operation with the controller turned on at $t = 100$.

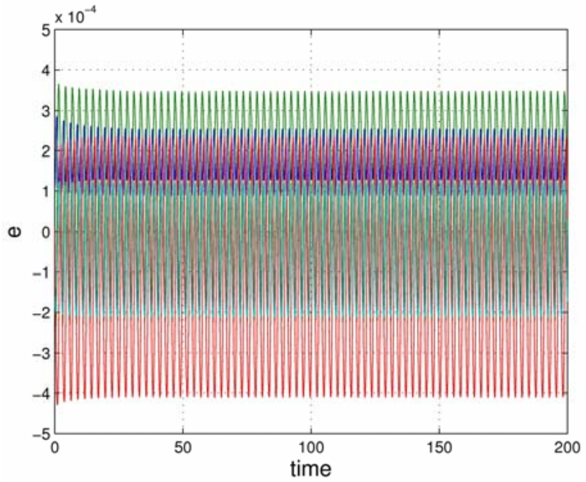

Fig. 9. Error between the reconstructed state $\hat{a}$ and the actual state $a$ for closed-loop operation with the controller turned on at $t = 100$.

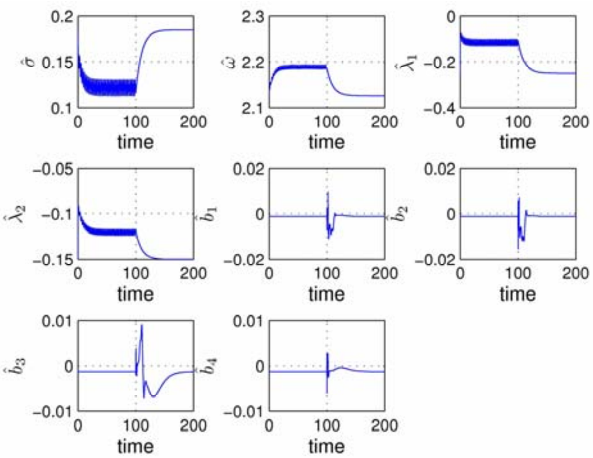

Fig. 10. Parameter vector $\hat{\theta}_L$ for closed-loop operation with the controller turned on at $t = 100$.

$\hat{\theta}_L$ estimated by the adaptation mechanism.It can be seen that there are significant variations in the estimated parameter values throughout the process. Nevertheless, since these parameters are estimated internally by the adaptation mechanism and hence available to the controller in real time, the controller can utilize the current value of the parameter vector $\hat{\theta}_L(t)$ to automatically schedule its matrices and hence succeeds in the desired stabilization despite considerable fluctuations in the parameter estimates $\hat{\theta}_L$.

## 8. CONCLUSIONS, DISCUSSIONS AND FUTURE WORKS

In this paper we considered a novel methodology for the control of nonlinear Galerkin models, which are an important class of nonlinear systems that arise frequently in the modeling of infinite dimensional systems. A linear parameter-varying (LPV) system is built to represent the nonlinear model, and an adaptation scheme is constructed to modify the parameter vector of the LPV system. The controller can then be designed on the LPV model, instead of the much more complicated nonlinear Galerkin system. An automatically scheduled $\mathscr{H}_\infty$ control technique is utilized for control design and it is shown that this controller does indeed stabilize the nonlinear Galerkin model, despite being designed on the simple LPV model. The approach proposed is illustrated with an example on cavity flow control, where the design is seen to produce satisfactory results.

The main contribution of the paper is to illustrate a novel approach to the control of nonlinear Galerkin systems through building an LPV model representing the nonlinear Galerkin model by adaptation techniques. Nonlinear Galerkin models are encountered frequently in fields such as partial differential equations (PDEs) and flow control problems; however, systematic analysis and design methodologies on these models have not been established. The method outlined in the paper reduces the complicated nonlinear Galerkin system to a simpler LPV system for control design purposes. Owing to their linear structure, LPV models are easier to analyze and control, thus many results and standard design techniques exist in literature dealing with LPV systems. In the paper we perform controller design using one such approach, namely the automatically-scheduled $\mathscr{H}_\infty$ design method. Once the LPV model is at hand, such a design can be performed straightforwardly using readily available routines in standard numerical computing software (e.g., MATLAB®. While the controller design is performed on the much simpler LPV model, we prove that it will achieve the desired stabilization when applied to the nonlinear Galerkin system as well. Such an approach can provide useful options and guidelines for researchers and engineers faced with the control of processes represented by Galerkin models.

Future research directions include using different adaptation laws and control techniques, and application of current and new approaches to other real-life flow control problems.

## APPENDIX: PROOF OF THEOREM 1

The proof relies on the concepts of dissipative systems and input-to-state stability (ISS) [38,39]. Since the controller is computed for the augmented LPV system using the approach in [32] as outlined in Section 4, the feedback system formed with this controller (18) and the LPV system (21)-(23) is a linear system satisfying $\| z \|_2 = \| \mathrm{col}(a,u) \|_2 < \gamma \| e \|_2$ for all $\hat{\theta}_L \in \Theta$. Hence the system (18), (21)-(23) is dissipative with a supply rate

$$q(e,a,u) = \gamma^2 \| e \|^2 - \| (a,u) \|^2 \tag{48}$$

and storage function

$$V_a(x_a) = x_a^T X_{cl} x_a \tag{49}$$

so that

$$V_a \le q(e,a,u) \tag{50}$$

is satisfied. Here, $X_{cl}$ is the solution of the matrix inequalities for the self-scheduled controller design, and $x_a := \mathrm{col}(\hat{a},\xi,\zeta)$ is the augmented state vector containing the states of the LPV plant, the input filter and the LPV controller. Note that

$$\underline{\alpha}_a(\| x_a \|) \le V_a(x_a) \le \overline{\alpha}_a(\| x_a \|), \tag{51}$$

where $\underline{\alpha}_a(r) := \lambda_{\min} r^2$, $\overline{\alpha}_a(r) := \lambda_{\max} r^2$ and $\lambda_{\min}$, $\lambda_{\max}$ are the smallest and largest eigenvalues of $X_{cl}$. Let us also define

$$V_t(e,\hat{\theta}_L) := \frac{1}{2} e^T e + \frac{1}{2} \tilde{\theta}_L^T \tilde{\theta}_L \tag{52}$$

and note that

$$\underline{\alpha}_t(\| \mathrm{col}(e,\tilde{\theta}_L) \|) \le V_t(e,\tilde{\theta}_L) \le \overline{\alpha}_t(\| (e,\tilde{\theta}_L) \|), \tag{53}$$

where $\underline{\alpha}_t(r) := \frac{1}{2} r^2$ and $\overline{\alpha}_t(r) := \frac{1}{2} r^2$. Consider now the entire system including the LPV plant, input filter, LPV controller, adaptation law and the nonlinear Galerkin model, which is an autonomous system (Fig. 2). Consider the state vector $x_e := \mathrm{col}(\hat{a},\xi,\zeta,e,\hat{\theta}_L)$ for the entire system. Note that the state of the Galerkin system is included implicity since $a = \hat{a} - e.$ Consider a candidate Lyapunov function

$$V(x_e) := V_a(\hat{a},\xi,\zeta) + V_t(e,\tilde{\theta}_L), \tag{54}$$

where $V_a$ is as defined in (49) and $V_t$ is as in (52). Note that

$$\underline{\alpha}_e(\| x_e \|) \le V(x_e) \le \overline{\alpha}_e(\| x_e \|), \tag{55}$$

where $\underline{\alpha}_e(r) = k_4 r^2$, $\overline{\alpha}_e(r) = k_5 r^2$ and $k_4$, $k_5$ are as in (33) and (34). Differentiating (54) along trajectories yields

$$\dot{V}(x_e) = \dot{V}_a(\hat{a},\xi,\zeta) + \dot{V}_t(e,\tilde{\theta}_L), \tag{56}$$

where we know that $\dot{V}_a$ satisfies (50). To obtain a bound for $\dot{V}_t$, note from (52) that

$$\begin{aligned}\dot{V}_t &= e^T \dot{e} + \tilde{\theta}_L^T \dot{\tilde{\theta}}_L \\ &= e^T \Big( \Phi_L(a,u)\hat{\theta}_L - ke - \Phi_L(a,u)\theta_L - \Phi_N(a,u)\theta_N \Big) \\ &\quad + \tilde{\theta}_L^T \Big( -\Phi_L^T(a,u)e - \Upsilon(\tilde{\theta}_L,a,u) - \Psi(e,\tilde{\theta}_L,a,u) \Big).\end{aligned}$$

Substituting (12)-(15) and simplifying

$$\begin{aligned}\dot{V}_t &= e^T \Phi_L(a,u)\hat{\theta}_L - e^T ke - e^T (\Phi_L(a,u)\theta_L \\ &\quad + \Phi_N(a,u)\theta_N) \\ &\quad - \hat{\theta}_L^T \Phi_L^T(a,u)e + \theta_c^T \Phi_L^T(a,u)e - \tilde{\theta}_L^T k_r \tilde{\theta}_L^* \|(a,u)\|^4 \\ &\quad - \tilde{\theta}_L^T k_s \tilde{\theta}_L^* \|e\|^2 - \tilde{\theta}_L^T \Psi(e,\tilde{\theta}_L,a,u) \\ &= -e^T ke - e^T (\Phi_L(a,u)\theta_L + \Phi_N(a,u)\theta_N \\ &\quad - \Phi_L(a,u)\theta_c)\end{aligned}$$

$$-\tilde{\theta}_L^T k_r \tilde{\theta}_L^* k_r \,\|\operatorname{col}(a,u)\|^4 - \tilde{\theta}_L^T k_s \tilde{\theta}_L^* \|e\|^2$$
$$-\tilde{\theta}_L^T \Psi(e,\tilde{\theta}_L,a,u)$$
$$= -k\|e\|^2 - e^T\big(La + L_{\text{in}}u + Q(a,a) + Q_{\text{ain}}(a,u)$$
$$+ Q_{\text{in}}(u,u) + L_c a + L_{c,\text{in}}u\big)$$
$$-k_r\tilde{\theta}_L^T \frac{\tilde{\theta}_L}{\|\tilde{\theta}_L\|^2}\|\operatorname{col}(a,u)\|^4$$
$$-k_s^T\tilde{\theta}_L^T \frac{\tilde{\theta}_L}{\|\tilde{\theta}_L\|^2}\|e\|^2 - \tilde{\theta}_L^T\Psi(e,\tilde{\theta}_L,a,u).$$

Taking norms and simplifying

$$\dot{V}_t \le -k\|e\|^2 + \|e\|\big(\|L\|\|a\| + \|L_{\text{in}}\|\|u\|$$
$$+\|Q\|\|a\|^2 + \|Q_{\text{ain}}\|\|a\|\|u\| + \|Q_{\text{in}}\|\|u\|^2 + \|L_c\|\|a\|$$
$$+\|L_{c,\text{in}}\|\|u\|\big) - k_r\|\operatorname{col}(a,u)\|^4 - k_s\|e\|^2$$
$$-\tilde{\theta}_L^T\Psi(e,\tilde{\theta}_L,a,u)$$
$$\le -k\|e\|^2 + k_6\|e\|\|\operatorname{col}(a,u)\|$$
$$+k_{65}\|e\|\|\operatorname{col}(a,u)\|$$
$$+k_7\|e\|\|\operatorname{col}(a,u)\|^2 - k_r\|\operatorname{col}(a,u)\|^4$$
$$-k_s\|e\|^2 - \tilde{\theta}_L^T\Psi(e,\tilde{\theta}_L,a,u),$$

where $k_6$, $k_{65}$ and $k_7$ are as given in (25), (35), (36). Using Young's inequality[4]

$$\dot{V}_t \le -k\|e\|^2 + k_6^2\|e\|^2 + k_{65}^2\|e\|^2 + \|\operatorname{col}(a,u)\|^2$$
$$+\|e\|^2 + k_7^2\|(a,u)\|^4 - k_r\|\operatorname{col}(a,u)\|^4 - k_s\|e\|^2$$
$$-\tilde{\theta}_L^T\Psi(e,\tilde{\theta}_L,a,u).$$

Collecting similar terms and using the fact that $\tilde{\theta}_L^T\Psi(e,\tilde{\theta}_L,a,u)\ge 0$ yields an upper bound for $\dot{V}_t$ as

$$\dot{V}_t \le -\big(k+k_s-k_6^2-k_{65}^2-1\big)\|e\|^2 + \|\operatorname{col}(a,u)\|^2$$
$$-\big(k_r-k_7^2\big)\|\operatorname{col}(a,u)\|^4 - \tilde{\theta}_L^T\Psi(e,\tilde{\theta}_L,a,u), \tag{57}$$

$$\dot{V}_t \le -\big(k+k_s-k_6^2-k_{65}^2-1\big)\|e\|^2 + \|\operatorname{col}(a,u)\|^2$$
$$-\big(k_r-k_7^2\big)\|\operatorname{col}(a,u)\|^4. \tag{58}$$

Substituting (50) and (58) into (56) yields

$$\dot{V}(x_e) = \dot{V}_a(\hat{a},\xi,\zeta) + \dot{V}_t(e,\hat{\theta}_L)$$
$$\le \gamma^2\|e\|^2 - \|\operatorname{col}(a,u)\|^2$$
$$-\big(k+k_s-k_6^2-k_{65}^2-1\big)\|e\|^2$$
$$+\|\operatorname{col}(a,u)\|^2 - \big(k_r-k_7^2\big)\|\operatorname{col}(a,u)\|^4 \tag{59}$$
$$\le -\big(k+k_s-k_6^2-k_{65}^2-\gamma^2-1\big)\|e\|^2$$
$$-\big(k_r-k_7^2\big)\|\operatorname{col}(a,u)\|^4,$$

[4]Let $x,y,\varepsilon\in\mathbb{R}_+$, then $xy \le \frac{x^2}{2\varepsilon} + \frac{\varepsilon y^2}{2}$.

which will be negative since from (24), (29) and (32) we know that $k > k_6^2+1$, $k_s > k_{65}^2+\gamma^2$ and $k_r > k_7^2$. Thus, all trajectories of the system are bounded and $e\to 0$, $a\to 0$ and $u\to 0$. The fact that $e=\hat{a}-a\to 0$ implies $\hat{a}\to a$, which states that the trajectories of the LPV system, whose parameter variations are controlled by the designed adaptation mechanism, will eventually approach those of the original nonlinear Galerkin model. The fact that $a\to 0$ states that the LPV control design based on the LPV plant is indeed successful in asymptotically stabilizing the origin of the nonlinear Galerkin model. This proves the first two statements of the theorem.

For the third statement, consider the error dynamics in (10) and parameter adaptation dynamics given by (11). We will first show that the system (10)-(11) is input-to-state stable (ISS), with $(a,u)$ viewed as the input to the system. We know that the system (10)-(11) is ISS if and only if it has an ISS-Lyapunov function. Consider $V_t(e,\hat{\theta}_L)$ given in (52) as a ISS-Lyapunov function candidate for the system. Differentiating $V_t$ along trajectories of the system yields the expression in (57), and from (29) it follows that

$$\dot{V}_t \le -\big(k+k_s-k_6^2-k_{65}^2-1\big)\|e\|^2 + \|\operatorname{col}(a,u)\|^2$$
$$-\tilde{\theta}_L^T\Psi(e,\tilde{\theta}_L,a,u). \tag{60}$$

Since (24) and (32) are satisfied, it holds that $k+k_s-k_6^2-k_{65}^2-1>0$. Also, if $\|\operatorname{col}(e,\tilde{\theta}_L)\|\ge k_x\|\operatorname{col}(a,u)\|$, then from (17) it follows that

$$\dot{V}_t \le -\big(k+k_s-k_6^2-k_{65}^2-1\big)\|e\|^2 + \|\operatorname{col}(a,u)\|^2$$
$$-k_d\|\tilde{\theta}_L\|^2,$$
$$\dot{V}_t \le -k_8\|\operatorname{col}(e,\tilde{\theta}_L)\|^2 + \|\operatorname{col}(a,u)\|^2,$$
$$\dot{V}_t \le -k_8\|\operatorname{col}(e,\tilde{\theta}_L)\|^2 + \frac{1}{k_x}\|\operatorname{col}(e,\tilde{\theta}_L)\|^2, \tag{61}$$
$$\dot{V}_t \le -\left(k_8-\frac{1}{k_x}\right)\|\operatorname{col}(e,\tilde{\theta}_L)\|^2,$$
$$\dot{V}_t \le -k_9\|\operatorname{col}(e,\tilde{\theta}_L)\|^2,$$

where $k_8 = \min(k+k_s-k_6^2-k_{65}^2-1, k_d)$ and $k_9 := k_8 - k_x^{-1}$. Note that $k_9>0$ as (31) and (32) hold. Then, if we define two $\mathcal{K}_\infty$ functions as $\chi(r) := k_x r$, $\alpha(r) := k_9 r^2$, it is true that

$$\|\operatorname{col}(e,\tilde{\theta}_L)\|\ge \chi\big(\|\operatorname{col}(a,u)\|\big) \Rightarrow \dot{V}_t \le -\alpha\big(\|\operatorname{col}(e,\tilde{\theta}_L)\|\big), \tag{62}$$

which is the definition of $V_t$ being an ISS-Lyapunov function. This shows that system (10)-(11) is ISS with $\operatorname{col}(a,u)$ regarded as the input. From the definition of ISS, this implies that there exists a class $\mathcal{KL}$ function $\beta$ and a class $\mathcal{K}$ function $\gamma$ such that

$$\begin{aligned}&\|\operatorname{col}(e(t),\tilde{\theta}_L(t))\|\\&\leq\max\left\{\beta(\operatorname{col}(e(0),\tilde{\theta}_L(0)),t),\gamma(\|\operatorname{col}(a,u)\|_\infty)\right\},\end{aligned}\tag{63}$$

where $\gamma$ can be shown to be of the form $\gamma=\underline{\alpha}_t^{-1}\circ\overline{\alpha}_t\circ\chi$ [5] Since $\beta\in\mathcal{KL}$, and $e(0)=0$, $\hat{\theta}_L(0)=0$ from (40)-(41), it holds that for all $t\geq 0$

$$\begin{aligned}&\|\operatorname{col}(e(t),\tilde{\theta}_L(t))\|\leq\gamma(\|\operatorname{col}(a,u)\|_\infty),\\&\|\operatorname{col}(e(t),\tilde{\theta}_L(t))\|\leq\underline{\alpha}_t^{-1}\circ\overline{\alpha}_t\circ\chi(\|\operatorname{col}(a,u)\|_\infty),\\&\|\operatorname{col}(e(t),\tilde{\theta}_L(t))\|\leq\underline{\alpha}_t^{-1}\circ\overline{\alpha}_t\circ\chi(\|\operatorname{col}(a,u)\|_\infty),\\&\|\operatorname{col}(e(t),\tilde{\theta}_L(t))\|\leq k_x\|\operatorname{col}(a,u)\|_\infty,\end{aligned}\tag{64}$$

and since $\|\tilde{\theta}_L(t))\|\leq\|\operatorname{col}(e(t),\tilde{\theta}_L(t))\|$ it follows that

$$\|\tilde{\theta}_L(t))\|\leq k_x\|\operatorname{col}(a,u)\|_\infty.\tag{65}$$

Recall that we know $\|\operatorname{col}(a,u)\|_\infty$ exists since the boundedness of all trajectories was established in Theorem 1. In fact, note that for the Lyapunov function $V$ defined as in (54), it can be seen from (59) that $\dot{V}\leq 0$ along trajectories. Hence, using (55), we can write

$$\begin{aligned}&\underline{\alpha}_e(\|x_e(t)\|)\leq V(x_e(t))\leq V(x_e(0))\leq\overline{\alpha}_e(\|x_e(0)\|)\\&k_4\|x_e(t)\|^2\leq k_5\|x_e(0)\|^2\\&\|x_e(t)\|^2\leq k_4^{-1}k_5\|x_e(0)\|^2.\end{aligned}\tag{66}$$

Note that

$$\begin{aligned}\|\operatorname{col}(a,u)\|^2=&\|\hat{a}-e\|^2+\|u\|^2\leq\|\hat{a}\|^2+\|e\|^2\\&+\|C_u\|^2\|\xi\|^2\leq k_{10}\|\operatorname{col}(\hat{a},e,\xi)\|^2,\end{aligned}$$

where

$k_{10}:=\max\{1,\|C_u\|^2\}$. Since $\|\operatorname{col}(\hat{a},e,\xi)\|\leq\|x_e\|$,

we have

$$\|\operatorname{col}(a,u)\|^2\leq k_{10}\|x_e\|^2.\tag{67}$$

Then from (66) and (67) we obtain

$$\|\operatorname{col}(a(t),u(t))\|^2\leq k_{10}k_4^{-1}k_5\|x_e(0)\|^2$$

and thus

$$\|\operatorname{col}(a,u)\|_\infty\leq\sqrt{k_{10}k_4^{-1}k_5}\,\|x_e(0)\|.\tag{68}$$

Substituting (68) into (65), and using (30) yields

$$\begin{aligned}&\|\tilde{\theta}_L(t))\|\leq k_x\sqrt{k_{10}k_4^{-1}k_5}\,\|x_e(0)\|,\\&\|\hat{\theta}_L(t)-\theta_c\|\leq k_x\sqrt{k_{10}k_4^{-1}k_5}\,k_{\mathrm{IC}},\\&\|\hat{\theta}_L(t)-\theta_c\|\leq\frac{\delta_\theta}{2}.\end{aligned}\tag{69}$$

[5]See for instance [39, p. 22].

Recall from (26), (27) and (28) that $\theta_c$ is the centroid of the $p$-dimensional box $\Theta$, and $\delta_\theta$ is the length of its shortest side. Hence (69) states that the parameter trajectories $\hat{\theta}_L(t)$ will be contained in a $p$-dimensional sphere $\mathcal{S}$ centered at the centroid of $\Theta$, whose radius is shorter that half the length of the shortest side of $\Theta$. Clearly $\mathcal{S}\subset\Theta$, hence the trajectories will be contained in $\Theta$, i.e., $\hat{\theta}_L(t)\in\Theta$. This proves the third statement of the theorem.

**Remark 4:** Note that the controller (18) designed for the LPV system (21)-(23) is also an LPV system itself. Although it was proven formally in Theorem 1 that this controller will stabilize the Galerkin system, one may still have some intuitive scepticism towards the result. It is possible to think that the LPV controller is essentially a linear controller and may not be powerful enough to stabilize the nonlinear Galerkin model (2). To address this concern, it should first be noted that the control input affects the system through three terms: $L_{\mathrm{in}}u$, $Q_{\mathrm{ain}}(a,u)$ and $Q_{\mathrm{in}}(u,u)$. The latter two terms are quadratic, and hence will enable a linear controller to generate terms that are quadratic in the state. An additional point to note is that the controller is not LTI, but LPV. That is, the parameters are allowed to vary with time, and how they will vary with time is determined by the designer. For instance, assume that we have an LPV control law of the form $u(t)=-K(t)a(t)$ where $K$ time varying. Suppose we design the time variation of the parameter as $K(t)=c_1+c_2a(t)^2$ where $c_1,c_2$ are positive numbers. Then the control law is essentially of the form $u(t)=-(c_1+c_2a(t)^2)a(t)=-c_1a(t)-c_2a(t)^3$ which is nonlinear in the state vector $a$. In summary, if needed, the LPV structure for the controller has the ability to generate control laws that are nonlinear in the state vector $a$.

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